

# Election Theory

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Primary Resource, Discrete Mathematics through Applications, W.H. Freeman and Company, 1999

## Unit Objectives

- Determine group rankings from individual rankings using the plurality, Borda, runoff, Hare (sequential runoff), Condorcet, pairwise, and approval voting methods.
- Identify violations of the majority, Condorcet, monotonicity, and independence of irrelevant alternatives criterions.
- Analyze group ranking methods for possible violations of Arrow's conditions.
- Analyze the fairness of distributed power in weighted-voting situations and, if necessary, change an unfair distribution so it is fairer.

### Day 1 “An Election Activity”

Instructor will provide each of the students with a slip of paper listing 5 popular soft drinks (all of the students will have the same list in the same order). The students will rate the soft drinks from 1-5 with 1 representing their first choice and 5 representing their last choice. After everyone has ranked the soft drinks, the students will display their rankings on the board. In small groups have the students come up with a way to organize the rankings of all the individuals into a single class ranking. Each group should decide and agree on one method and upon completion present their ranking and method to the other groups. All of the data collected on the soft drink preferences should be saved for future use. Have the students complete exercises 1-8 on pg 3 in which they are asked to analyze the various group ratings for first, second, third, fourth, and fifth and compare these results with their own group results. Exercise 6 has the students write a description of their group method and exercise 8 has the students consider the issue of ties and tie breakers. Instructor can lead the students through exercise 9 and discuss a preference schedule followed by exercise 10 where the students are asked to reflect on our U.S. election system and whether or not allowing voters to vote by ranking candidates would be a beneficial practice. Have the students complete exercises 11-13 on pg 5-6 which introduce the students to the number of possible schedules with four, five, and six choices as well as recurrence relations and completing tables for given recurrence relations.

### Day 2 “Group Ranking Methods”

Instructor will provide discussion and examples of the plurality method (clarification of plurality vs. majority), Borda method, runoff method, Hare method (sequential runoff method), and pairwise comparison method with a guided practice after each to work on in small groups. Using the previously collected data on the 5 popular soft drinks, have the students explore and explain which soft drink is the plurality winner and whether or not this soft drink is also a majority winner, which is the Borda winner, runoff winner, Hare winner and pairwise comparison winner. Complete exercises 5-11 on pg 12-14 which offer various election results for the students to practice the various methods and rankings. Exercise 11 builds on the ideas from the previous day's homework problems 11-13 by having them discover a recurrence relation.

### Day 3 “Algorithms and Matrices”

Instructor will provide the class with an example algorithm, explaining what an algorithm does. Following the given example, have the students write out the algorithms that they have been using for the Hare method, Borda method, and pairwise comparison method. Have the students do this in

small groups with each group writing out the algorithms for 2 of the methods. Have the groups present their algorithms and decide on a final written algorithm for each of the methods. Introduce the idea of using matrices with the Borda method and discuss any needed background information on matrix multiplication. Using guided practice and an additional example, have the students determine the winner using matrix multiplication with the Borda method.

#### **Day 4 “Condorcet Candidates and Paradoxes”**

Instructor will provide background information on Marquis de Condorcet, the concept of a Condorcet winner and situations where the Condorcet method does not produce a winner. A class discussion with examples should follow on the “fairness” of the various methods in considering the Condorcet criterion. Have the students work through exercises 1-7 on pg 20-21 which provide situations exploring the Condorcet method, pairwise voting, the runoff method, and the idea of “voting insincerely” which leads to the monotonicity criterion. The monotonicity criterion can also be introduced at this time with exercise 8 on page 22. The independence of irrelevant alternatives criterion can be introduced with exercise 9 followed by exercise 10 where the students are expected to write a brief summary of the 6 methods of achieving a group ranking (plurality, Borda, runoff, Hare, pairwise and Condorcet). As a mid-unit check, have the students provide one example of how each method could lead to results that are not “fair” specifying the method used and why it is not fair. Exercises 11 and 12 should be assigned to build on their recurrence knowledge and as a preview for diagram representation and terminology such as vertices, edges and possible combinations of comparisons.

#### **Day 5 “Arrow’s Conditions and Approval Voting”**

Instructor will provide background information on Kenneth Arrow and the five fairness conditions named after him known as “Arrow’s conditions” as listed in the primary resource (pg 26-27).

1. Nondictatorship: The preferences of a single individual should not become the group ranking without considering the preferences of the others.
2. Individual Sovereignty: Each individual should be allowed to order the choices in any way and to indicate ties.
3. Unanimity: If every individual prefers one choice to another, then the group ranking should do the same. (In other words, if every voter ranks A higher than B, then the final ranking should place A higher than B.)
4. Freedom from Irrelevant Alternatives: The winning choice should still win if one of the other choices is removed. (The choice that is removed is known as an irrelevant alternative.)
5. Uniqueness of the Group Ranking: The method of producing the group ranking should give the same result whenever it is applied to a given set of preferences. The group ranking should also be transitive.

Following this background information, the students should complete exercises 1-7 in which the students will study a number of group-ranking methods to see if they violate Arrow’s conditions and will also lead them to discover that any group ranking method will violate at least one of Arrow’s conditions. As a class, discuss student ideas on how to resolve this or how to make it better.

Following the discussion, introduce the approval voting system (voters may vote for as many choices as they like, but they do not rank them. The voter selects all of the candidates they would approve of). Have the class conduct an approval vote for the soft drinks from the Day 1 activity following with exercise 8 on pg 28, which has the students determine a complete group ranking and comparing this result with the earlier results using the plurality and Borda methods. Exercises 9-11 on pg 28-29 may be used to explore the fairness of approval voting. To further build on recursive relations, the students may complete exercises 12-17 on pg 29-31 which explore the number of ways in which you can vote under approval voting and the patterns that emerge in the various ways of voting when the approval system is used.

### **Day 6 “Weighted Voting and Voting Power”**

Instructor will provide the students with the necessary terminology such as weighted voting (when some members of the voting body have more votes than others have), coalitions (a collection of voters), winning coalitions (a coalition with enough votes to pass an issue), and power index (a measure of the power of a member of a voting body). Because there are various ways to measure the power of a member of a voting body, it would be helpful to provide the students with an algorithm to use such as the following, “A Power Index Algorithm”, as listed in the primary resource.

1. List all coalitions of voters that are winning coalitions.
2. Select any voter, and record a 0 for that voter’s power index.
3. From the list in step 1, select a coalition of which the voter selected in step 2 is a member. Subtract the number of votes the voter has from the coalition’s total. If the result is less than the number of votes required to pass an issue, add 1 to the voter’s power index.
4. Repeat step 3 until you have checked all coalitions for which the voter chosen in step 2 is a member.
5. Repeat steps 2-4 until all voters have been checked.

Exercises 1-10 on pg 35-36 should be completed in order to provide the students with adequate experience listing all possible coalitions and all winning coalitions, determining power indexes, identifying members that have all the power (*dictators*) and members that have no power (*dummies*) and exploring new power distributions in situations with three voters. Exercises 6 and 7 deal specifically with writing recurrence relations and listing all possible coalitions. Exercise 10 provides an extension on the winning coalition situation by discussing a “minimal” winning coalition in which all the voters are essential.

### **Day 7 “Class Preference Schedules for Individual Issues of Current Interest”**

Have each student select an issue of current interest in the community that involves more than two choices. Each student should then have all the members of the class vote on the issue using a preference schedule and providing their class label of Freshman, Sophomore, Junior, and Senior. Following the collection of the ballots, have the students compile the results and determine the winning solution by using the 6 voting methods as listed in Day 4 of this unit. Each method should be examined for “fairness” using Arrow’s conditions. As an extension, the students should use their results from the vote to consider the class label of each voter with a senior vote worth 8, junior vote worth 6, sophomore vote worth 4 and a freshman vote worth 2 and run the Power Index Algorithm.

### **Day 8 and 9 “Student Presentations”**

Have the students present on the data they have gathered and analyzed. After all students have presented, a class discussion should follow on the presentations and the activity as it relates to the students’ lives.

#### **Possible Extensions**

- \*Proportional representation
- Party-list system
- Mixed member system
- Preference vote system

#### **Blackline Masters (from primary resource)**

- \*Day 1
- 1.1
- \*Day 2
- 1.2, 1.3, 1.4 (exercise 7)

\*Day 4

1.5 (exercises 3 and 4), 1.6 (exercises 7 and 8), 1.7 (exercise 9)

\*Day 5

1.8, 1.9, 1.10 (exercise 11)

## **Election Theory**

**Content Standard:** Data Analysis, Statistics, and Probability

**Level:** 9-12

### ***Specific Statement(s) from the Standard:***

Compare outcomes of voting methods such as majority, plurality, ranked by preference, run-off and pair-wise comparison. (Minnesota Academic Standards Mathematics grades 9-12)

### ***Product(s):***

Student presentation on issue of current interest in the community and winning solution(s) determined by 6 voting methods using Arrow's conditions for analysis of fairness and discussing how the results would change if weighted voting were used.

### ***Task Description:***

Each student will select an issue of current interest in the community that involves more than two choices. Each member of the class will vote by writing a preference schedule (including their class label of freshman, sophomore, junior, or senior) results of which will be compiled and the winning solution determined by 6 voting methods. With the understanding that Arrow's conditions result in impossibility, the student must discuss the fairness of each voting method with respect to Arrow's conditions. Each student will then consider the class label of each voter with a senior vote worth 8, junior vote worth 6, sophomore vote worth 4 and a freshman vote worth 2, which will be used to run the Power Index Algorithm. The students will prepare a presentation on the data they have gathered and analyzed.

### ***Special Notes:***

Depending on time constraints, the instructor may choose to eliminate the topics from day 6 and modify the assessment accordingly.

## FEEDBACK CHECKLIST FOR ELECTION THEORY

The purpose of the checklist is to provide feedback to the student about his/her work relative to the content standard. Have the standard available for reference.

Y=Yes

N=Needs Improvement

<u>Student</u>		<u>Teacher</u>
_____	An issue of current interest that involves more than two choices has been selected	_____
_____	Preference schedules have been collected and compiled	_____
_____	Results of the vote have been determined using 6 voting methods	_____
_____	Violations of Arrow's conditions have been considered for all 6 voting methods	_____
_____	Fairness of each of the 6 voting methods have been addressed	_____
_____	Using the data collected, the student has analyzed how weighted voting would affect the results of the voting system	_____

**Overall Comments** (information about student progress, quality of the work, next steps for teacher and student, needed adjustments in the teaching and learning processes, and problems to be addressed):

Voter Profile

# of voters: preference:	8	5	6	7
1 <sup>st</sup> choice	A	B	C	D
2 <sup>nd</sup> choice	B	C	B	B
3 <sup>rd</sup> choice	C	D	D	C
4 <sup>th</sup> choice	D	A	A	A