Discrete Math Project

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June 23, 2006
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Computers have changed our culture in fundamental ways, including the way we learn mathematics, the way we do mathematics, the kinds of problems we can consider. Even our imaginations, our creative visions, and our sense of what is possible have been altered. One way to think about computers is as discrete machines, capable of dealing only with finite information. Discrete mathematics, with its many real-world applications and its close ties to computer science, has grown rapidly over the last thirty years. In the words of John Dossey, it is the "math for our time." (Dossey quoted in Kenney, 1991).

Applications of discrete mathematics are found in a variety of settings, including project management, communication networks, systems analysis, social decision making, population growth, and finance. Discrete mathematics is used to design efficient computer networks, optimally assign frequencies to cellular phones, track pollution, fairly rank competitors in a tournament, accurately represent public opinion in political elections, efficiently schedule large projects, plan optimal routes, and solve many other problems, both applied and abstract. These applications oblige us to provide students the knowledge and skills of discrete mathematics to prepare them for life work in the twenty-first century.

Discrete mathematics is concerned with finite processes and phenomena. It involves the study of objects and ideas that can be divided into 'separate' or 'discontinuous' parts. While discrete mathematics is sometimes contrasted with calculus, which focuses on infinite processes and continuous phenomena, it is more a complement to calculus than a competitor.

The concepts and techniques of discrete mathematics can be used to model and solve problems involving enumeration (determining a count), decision making in finite settings, relationships among a finite number of elements, and sequential change. It is used to investigate settings in which functions are defined on discontinuous sets of numbers, such as the positive integers (Dossey in Kenney, 1991, p. 1).

Given the relative novelty of discrete mathematics to average citizens as well as many teachers of mathematics, it may be more helpful to consider some characteristics of discrete mathematics than to examine its definitions.

Problems in discrete mathematics can be classified into three broad categories:

1. Existence problems deal with whether or not a solution exists for a given problem.

A familiar context for this type of situation is the eighteenth-century problem that intrigued the Swiss mathematician Leonard Euler (1707-1783). In the 1700s, seven bridges connected two islands in the river to the rest of the city of Königsberg (see Figure 1).

![Figure 1. Representation of the seven bridges of Königsberg (now known as Kaliningrad, Russia)](image)
Is it possible to walk through the city by crossing each bridge exactly once and return to the original starting point? Using a vertex-edge graph in which the vertices represented the landmasses of the city and the edges represented the bridges, Euler found that there was no such walk possible. His investigation of this problem, however, led Euler to make a number of generalizations about the traceability of vertex-edge graphs.

To learn more about what Euler discovered, try to traverse the following graphs without lifting your pencil or tracing the edges more than once. When can you draw the figures without retracing any edge and still end up at your starting point? When can you draw the figure without retracing any edge but end up at a point different from where you started? When can you NOT draw the figure without retracing an edge? (Crisler et al., 1994, pp. 174-5).

2. Counting problems explore how many solutions may exist for a given problem.

A familiar application is the number of phone numbers that can exist for a given area code. A seven-digit telephone number cannot begin with a 0 or a 1. A common strategy is to apply the multiplication principle: multiply the number of possible choices for the first digit, second digit, etc.: 

8 x 10 x 10 x 10 x 10 x 10 x 10. There are 8,000,000 possible telephone numbers for a given area code.

Prior to 1996, Minnesota was served by three area codes: 218, 612, and 507. In 1996, the 320 area code was added. What information would you need to know in order to predict whether or not the state will need a new area code in the near future?

3. Optimization problems focus on finding a “best” solution to a particular problem, “best” being defined by the context of the problem—the most efficient method, or the shortest path, or the fairest decision, etc.

Fair division conflicts are common. One contest is voting apportionment schemes. Central High School has 464 sophomores, 240 juniors and 196 seniors. The problem involves dividing the 20 seats on the student council among the three classes. An ideal ratio (total population divided by the number of seats) results in 45 students per seat. The calculated quotas for each class (the class size divided by the ideal ratio) result in decimal values. That presents a dilemma, as a single seat cannot be split to give part of it to the seniors, part of it to the juniors, and part of it to the sophomores. How could the 20 seats be distributed fairly? (Crisler et al., 1994, pp. 59-60)

These problems illustrate the characteristic ways of thinking in discrete mathematics that can be developed over the entire mathematics curriculum. The basic questions that should be asked at every level are:

1. Is there a solution to this problem?
2. How can we solve this problem? How many solutions are there?
3. Which of these solutions is the “best” within the context of the problem?
Discrete mathematics is especially rich in the variety of applications it can treat. Some typical examples include:

**routing problems**—design an efficient plan for city-wide snow removal

**scheduling problems**—design an optimal schedule for legislative subcommittee meetings

**matching problems**—fill necessary jobs with capable applicants at minimal cost

**sorting problems**—describe an efficient method for alphabetizing 100,000 names

**searching problems**—describe a method for locating a particular item in a data base

There are some content areas that are typically identified with the field of discrete mathematics. These include:

**combinatorics**—the application of systematic counting techniques

**graph theory**—the use of vertex-edge diagrams to study relationships among a finite number of elements

**game theory**—the mathematics of voting, fair division, apportionment, and cooperation and competition

**recursion**—the method of describing sequential change by indicating how the next stage of a process is determined by previous stages

**algorithmic thinking**—the development and analysis of a rule system to solve a problem or a class of problems

This last notion, algorithmic thinking, is a particularly important concept in discrete mathematics. Problems precede algorithms, but once the problem has been stated, the focus is on how it might be solved. Our interest in algorithms is in their development and analysis; computers can usually carry out the rote steps of algorithms. Examples of algorithmic thinking include whether all the proposed solution procedures are correct and which are most efficient. It is in the design of algorithms that new insights into the original problem are often found.

Discrete mathematics is accessible to students at all levels. Arithmetic offers a fertile field for interesting problems in discrete mathematics. Many practical everyday problems can be modeled as graphs. Almost any puzzle or challenging problem, even (and especially) those of a recreational nature, will involve discrete mathematics in some form. There is a growing literature of excellent materials that can be used to promote discrete thinking. (See the Sample Problems, the Sample Tasks, and the Teaching Resources in this section.)

There are workable and practical ways to include discrete mathematics in an already overcrowded curriculum (Hart, 1991, pp. 76-77).

- Many topics in discrete mathematics, including matrices, counting techniques, induction, sets, and sequences overlap with other content strands. A teacher can emphasize these topics that do “double duty” in the curriculum.

- The tools and techniques of discrete mathematics can be used to approach traditional mathematics in new ways. For example, recursive formulas can represent sequential change, vertex-edge graphs can model relationships in mathematical problems, or matrices can be used to solve systems of linear equations.
• Short units can be taught on discrete topics, such as graph theory or game theory.
• Mathematics courses can be integrated in such a way that discrete mathematics occurs amidst algebra, trigonometry, geometry, statistics and probability topics. Many new standards-based curriculum projects, developed with sponsorship from the National Science Foundation, take this approach to content organization, and hold promise for delivering the Minnesota Graduation Standards in mathematics in efficient and meaningful ways.

Discrete mathematics is both important and relevant to many real-world situations. It is, in fact, the mathematics used by many decision-makers in our society, including workers in health care, transportation, telecommunications, and a variety of government agencies. Introducing topics from discrete mathematics thus serves to broaden students’ knowledge of the range of mathematics while making the “school to work” connection.

The inclusion of discrete mathematics in the K-12 curriculum has other payoffs for teachers and students:

1. Discrete mathematics is full of unsolved problems and unique strategies. A focus on discrete mathematics reinforces the central theme of problem solving in mathematics education while communicating to students the contemporary and dynamic nature of mathematics. It increases students’ understanding of what it means to DO mathematics by encouraging them to formulate and test conjectures.

2. The study of discrete mathematics also has the potential to promote students’ critical thinking, mathematical reasoning, and visualization skills while making important connections between mathematics and unique problem situations. Discrete mathematics often utilizes geometric ideas in ways that complement symbolic manipulation, providing students with multiple ways to think about and approach problems.

3. Discrete mathematics does not have extensive prerequisites, yet it poses challenging problems to all students. These challenges are engaging and accessible, and can broaden and enrich the other content strands. Discrete mathematics has the potential to stimulate greater interest in mathematics among students of all abilities at all grade levels.

4. Discrete mathematics has the potential to provide students with an array of new and powerful models for thinking about and doing mathematics, including vertex-edge graphs and matrices.

We will need to renew ourselves as learners of mathematics to give adequate time and attention to this relatively recent field of discrete mathematics. The Curriculum and Evaluation Standards for School Mathematics of the National Council of Teachers of Mathematics and the Minnesota Graduation Standards recommend that our teaching style should reflect a problem solving approach that permeates the entire mathematics program. Discrete mathematics provides rich and motivating contexts and challenges in which both teachers and students can become engaged, thereby increasing the chances that both groups will continue their study of mathematics and improve their problem solving skills.
Chromatic Chemicals

Objectives: Students will
• use the QUEST problem-solving algorithm
• determine the chromatic number* of a graph representing a problem

Math Topic: Discrete math

Lead-up topics:
• Graph coloring (maps)
• Euler paths and circuits
• Problems involving scheduling conflicts

Materials:
• projection materials
  1. Problem Solving QUEST
  2. Chemical Reaction Table
  3. The QUEST, The Problem
• handouts
  1. Chemical Reaction Table (copies as needed)
  2. The QUEST, The Problem (1 per student)

Introduction:
“What are twelve items you would buy in a grocery store?”
“How many bags would you use to carry them?”

“Instead of grocery items, we’re going to consider chemicals at your workplace.”
“How would you store twelve chemicals, some of which are unsafe when mixed?”
“We’re going on a quest for the answer, a Problem Solving QUEST.”
Display Problem Solving QUEST.

*Chromatic number – “The smallest number of colors needed to color a graph so that no edge connects vertices of the same color …”

**Instruction:**

Hand out, display, and discuss the chemical reaction table.

Hand out and display The QUEST, The Problem.

Elicit or provide the following responses to the questions and objectives of QUEST:

- **Q** – How many cabinets (or other storage locations) are needed?
- **U** – Chemical reaction table
- **E** – Chemicals that are relatively safe when mixed, and those unsafe when mixed
- **S** – Draw a conflict graph, color the vertices, determine the chromatic number
- **T** – Confirm that chemicals unsafe when mixed are all stored separately

Students’ solutions should be similar to the following graph:

![Graph](image-url)

**Extensions:**

- Discuss the properties and uses of the chemicals.
- Hand out and discuss Material Safety Data Sheets.
- Have students determine the number and type of storage locations for tools and materials at their workplaces.

Problem Solving QUEST

Question the situation

Useful information

Elements to consider

Strategy of action

Test the answer
## Chemical Reaction Table

### Chemicals

1. Toluene
2. Acetone
3. Phosphoric acid
4. Sulfuric acid
5. Potassium cyanide
6. Sodium hydroxide
7. Dimethyl hydrazine
8. Dinitrogen tetroxide
9. Chromic anhydride
10. Nitrogen
11. Chlorine
12. Potassium dichromate

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<td>S</td>
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S: relatively safe when mixed  
U: unsafe when mixed

<table>
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<tr>
<th><strong>The QUEST</strong></th>
<th><strong>The Problem</strong></th>
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<tbody>
<tr>
<td><strong>Question the situation</strong></td>
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<tr>
<td>Define the problem.</td>
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<tr>
<td>What is the question to be</td>
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<td>answered?</td>
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<tr>
<td><strong>Useful information</strong></td>
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<td>Determine the facts.</td>
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<td>What is known and needed to</td>
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<td>solve the problem?</td>
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<td><strong>Elements to consider</strong></td>
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<tr>
<td>Describe the situation.</td>
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<td>Who and/or what is</td>
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<td>involved?</td>
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<td><strong>Strategy of action</strong></td>
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<td>Devise and use a plan of</td>
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<td>operations.</td>
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<td>What strategies apply to the</td>
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<td>situation?</td>
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<td><strong>Test the answer</strong></td>
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<td>Check the solution.</td>
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<tr>
<td>Does the answer make</td>
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<td>sense?</td>
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<tr>
<td>• Solution supported</td>
<td></td>
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<tr>
<td>• Audience agrees</td>
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“Coloring Problems”
Graphs to avoid conflicts!

The table shows clubs at our high school and it shows the students who hold offices in the clubs. Here’s the problem: any clubs who have the same officers have to meet on different weeknights so officers can always attend all their meetings. Office holders are marked with X. Each club meets once a week. What is the minimum number of days needed?

<table>
<thead>
<tr>
<th>Student</th>
<th>Math</th>
<th>Honor</th>
<th>Science</th>
<th>Art</th>
<th>Pep</th>
<th>Spanish</th>
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<td>X</td>
<td>X</td>
<td>X</td>
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<td>Larry</td>
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<td>-</td>
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</table>

Construct a graph in which the vertices of the graph represent the clubs. The edges represent clubs that share at least one officer. So Math and Pep clubs get an edge because Ole is an officer in both. This represents a potential conflict and those two clubs cannot meet on the same night.

Begin labeling the vertices of the graph with the days of the week. For example, label Math Monday. Any club that has an edge with math cannot meet on Monday!

Since no officer serves both Math and Spanish clubs, Spanish club could meet on Monday, too. Label Spanish Monday.

Label the Honor Club Tuesday. Either the Pep or Art Club could also be labeled Tuesday; note that you have to make a choice between the two clubs. Just do it and label one of the choices with Tuesday.

Label the other choice as Wednesday. Note that the Science club can also be labeled Wednesday.

We have now scheduled everyone in the minimal number of days. There may be other patterns that will work with the same number of days, so our solution may not be unique!

Carpool Problem:
Mr. Sarles wanted to take 8 students from his class on a field trip. Here is a listing of the students who are not compatible so they cannot ride in the same car. Assuming he has enough cars and drivers, what is the minimum number of vehicles he will need so there are no fights?

Conflicting pairs: AB, AC, AF, BG, BH, BC, CD, CE, CH, DE, EG, EF.

Draw the graph.
(Answer: 3 vehicles. Put A in the red car. Try B in the green car and F and C in the blue car. Now H can go in the red car (but not in the blue or green.) D can go in the red car, too….)
**Devise an Experiment**

Goals:
To use theory and practice and compare results
To tabulate data and evaluate it
To see how sample size influences results

Step 1:
Given a bag of ten cards numbered from zero to ten, choose a card, write down the number in a “hundreds place;” put the card back. Choose a second card; write the number in the “tens place” and put the card back. Pick the third card, write the number in the “ones place,” and put the card back. You get a three-digit number. Then do the same method to pick three more for a second three-digit number.
Ask: What are the chances the sum of the two numbers will be even? Make a prediction.
Discuss: A trial is doing the task once.

Step 2:
Pick a card from the bag, record the number, then put the card back in the bag. Repeat this step again, and then a third time so you make a three-digit number. (If you choose a zero where the hundreds digit goes, keep it as a placeholder.) Then pick another and record it, a second and record it, and a third and record it, making another three-digit number.
Ask: What are the chances the sum is even?
Ask: What are the chances the product is even?
Discuss: When is the product of two numbers even? When is it odd? How does this idea help you predict what will happen?
Discuss. Predict what the result will be when the whole class shares results.
Discuss: Relative frequency is the number of times you get a desired result divided by your total number of trials. Do you think your (individual or team) results will be “right on?” Do you think your class’s results will be “right on?”

Step 3:
Use Step two as your first “data point” for your experiment. Repeat this step until you have made nine pairs of three-digit numbers. Record your results, listing the number of pairs you multiplied, the number that had even products, and the ratio of even products to total products.

Step 4:
Each class member or team should record results and make a larger table for the whole class.
Ask: What is the ratio of even products to total products?
Ask: Do our result seem reasonable?
Ask: Does your result change when you compare a small group to the whole
Predict Frequency

Goals:
Inspect sample spaces for things that are easy to predict.
Use experimental data to develop a probability.
Compare and contrast the experimental and theoretical probability.

Step 1: Roll a six-sided pencil on a flat surface. Predict the chances that it will land with the brand name up. Repeat the experiment 18 times and write about your results. Have the class put all their results together, and discuss how accurate your predictions were. What information could you use to improve your predictions?

Step 2: Roll a number cube. Predict the chances that the number 6 face will land upward. Repeat your experiment 18 times and write about your results. Have the class put all their results together and discuss how accurate your predictions were. What information could you use to improve your predictions?

Step 3: Toss a paper cup and see if it lands on its side. Predict the chances a paper cup will land on its side when tossed upward and allowed to drop. Repeat your experiment 18 times and write about your results. Have your class put all their results together and discuss how accurate your predictions were. What information could you use to improve your predictions?
**Probability: Tossing a Number Cube, etc.**

Goals:
- Develop vocabulary.
- Chart sample spaces.
- Calculate probability.
- Discover when $P=0$ and when $P=1$.

The result you get from tossing a number cube is called an outcome. If you roll a number cube and the 6 is face up, your outcome is a six.

When you roll a number cube, you are just as likely to get a 1 as a 2 as a 3 as a 4 as a 5 as a 6. When each outcome is just as likely, the outcomes are called “equally likely outcomes.”

You can figure out all the outcomes you could get if you rolled a red number cube and a green number cube. You might get a red 4 and a green 5. Then again, you might get a red 2 and a green 6. If you list all the different answers that could come up, you have listed the sample space.

Any part of a sample space is called an event. It could be one, more than one, or none!

The probability that an event will occur, $P(E)$, is the ratio of the number of favorable outcomes to the number of possible outcomes in the sample space. For example, the probability of tossing a 3 on a number cube is one out of six.

The probability when an event is impossible is zero.

What is the probability when an event is certain?

Step 1: List all the outcomes for the toss of one number cube.


Step 3: Find other examples, such as:
   a. Toss a 5 or 6
   b. Toss an even number
   c. Toss an 8
   d. Toss an number less than 7

Step 4: Give the sample spaces for several models, such as:
   a. Flip of a coin
   b. Toss of a cube with letters of the alphabet on the faces
   c. Pick a card from a deck
   d. Pick a number from one to ten
   e. Draw a certain color from a bag of ten marbles of various colors (known amounts)

Step 5: List the favorable outcomes and find the probability of each event. Classify whether each is certain, impossible or in between. Use examples such as
a. Spin a vowel on a spinner
b. Draw a blue cube from the bag. (Check the contents first.)
c. Pick a multiple of 4 from a deck of cards (include queen)
d. Pick a joker from a deck of 52 cards.
e. Pick a prime number from a deck of 52 cards.
f. Spin a letter from your first name on a spinner.
g. Flip a head on a coin.
h. Spin a W on a spinner.
i. Spin an odd number on a spinner.
j. Pick a factor of 60 from a deck of cards.
k. Flip a head or a tail on a coin.
l. Pick a card from a deck to get a multiple of 5 or 12.
Probability Simulation

A random number generator can simulate a probability experiment. From the simulation, you can calculate experimental probabilities. Repeating a simulation may result in different probabilities since the numbers generated are different each time.

Example Generate 30 random numbers from 1 to 6, simulating 30 rolls of a number cube using a TI-73 graphing calculator.

- Access the random number generator.
- Enter 1 as a lower bound and 6 as an upper bound for 30 trials.

**KEYSTROKES:** 2 1 6 30

A set of 30 numbers ranging from 1 to 6 appears. Use the right arrow key to see the next number in the set. Record all 30 numbers, as a column, on a separate sheet of paper.

Exercises 1–4

1. Record how often each number on the number cube appeared.
   a. Find the experimental probability of each number.
   b. Compare the experimental probabilities with the theoretical probabilities.

2. Repeat the simulation of rolling a number cube 30 times. Record this second set of numbers in a column next to the first set of numbers. Each pair of 30 numbers represents a roll of two number cubes. Find the sum for each of the 30 pairs of rolls.
   a. Find the experimental probability of each sum.
   b. Compare the experimental probability with the theoretical probabilities.

3. Design an experiment to simulate 30 spins of a spinner that has equal sections colored red, white, and blue.
   a. Find the experimental probability of each color.
   b. Compare the experimental probabilities with the theoretical probabilities.

4. Suppose you play a game where there are three containers, each with ten balls numbered 0 to 9. Pick three numbers and then use the random number generator to simulate the game. Score 2 points if one number matches, 16 points if two numbers match, and 32 points if all three numbers match. Note: numbers can appear more than once.
   a. Play the game if the order of your numbers does not matter. Total your score for 10 simulations.
   b. Now play the game if the order of the numbers does matter. Total your score for 10 simulations.
   c. With which game rules did you score more points?

Graphing Calculator Investigation TI-73 Probability Simulation 315 A Follow-Up of Lesson 6-9

www.pre-alg.com/other_calculator_keystrokes
**Election Activity: Choose your favorite candidates**

Based on lesson 1 in *Discrete Mathematics Through Applications* by Crissler, et. al.

Have students choose a topic like one of the following, and get four or five “candidates.”
- Favorite movie
- Favorite CD
- Favorite soda pop
- Hypothetical homecoming queen candidates
- Favorite color (from a selected set)

Once the topic is chosen and there are four or five candidates, each student creates a ballot so that the candidates are all listed in the same order.

Each student votes in the following manner: put a 1 by the most favorable candidate, a 2 by the second-most favorable candidate, a 3 by the third-most favorable candidate, etc. until each candidate has a ranking.

Make a chart on the board and record rankings by all voters. Each voter writes in his/her own column. (Then you can see how many students rated candidate A first, second, third, fourth, etc., and how many rated B as first, second, etc.) Note common patterns.

In groups of three, students should figure out a method to choose a winner, second place, third place, etc. Each group will explain their choice and their method and reasons.

Discuss results.
- Did groups get different rankings?
- What methods seemed fair and reasonable? Were any methods not reasonable?
- Which method seems most agreeable to most class members?
- Did you get any ties? How did you deal with them?

Work to come up with a “consensus” which will represent our class’s vote (rank of all candidates, first to last) as we enter the pool with other classes. Get the data (or make it up) for 3 or 4 other classes. Each class’s information will later be made into a “preference schedule” so votes of several groups can be compared. Chart the data.

Discuss the **Plurality Method**. What are the pros and cons?

Have each group use the method they devised. How do class results compare? Did any groups revise their plan?

Refer to a discrete math text to get rules on the **Borda method**. (This “weights” the candidates by their ranks. Given five candidates, go group by group. Multiply the number of voters for the
first-ranked candidate by 5, the second-ranked candidate by 4 points, and on down. Then sum Candidate A’s points from all the groups, candidate B’s votes from all the groups, etc. Top sum wins.) Demonstrate part of it and let the students finish the pattern. How do these results compare to previous results? Would any class members like to revise their approach?

Show how to format a preference schedule. Have each group write some. (For each group of voters, draw an upward-pointing arrow. For a given group, write the name of the top candidate on top, followed by the second-most favored, etc, to the lowest ranked candidate at the bottom. Write the number of voters for that group below the arrow. If six groups voted, there would be six arrows.)

Use the “Runoff Method” by following these steps:
Write preference schedules for the candidates.
Determine the number of votes for first place for each candidate.
Eliminate all but the two candidates with the highest number of first rankings. Go through the preference schedules and draw a line though everyone but the two top candidates.
Rewrite the preference schedules with just the two top candidates.
Choose the higher candidate on each schedule and award the votes to that candidate; cross of the loser on each preference list.
Tally the votes for each candidate. Highest total wins. Discuss results.

Try the Sequential Runoff Method.
Candidates will be eliminated one at a time. Re-copy the original preference schedules.
Pick the candidate who has the fewest first-place votes. Cross that candidate off all preference schedules. Rewrite the schedules and figure out which of the remaining candidates has fewer first-place votes. Cross off that candidate on all schedules. Keep going until only one remains.
Discuss results.
Apportionment

Adapted from Crisler, et. al., Discrete Mathematics through Applications.

What if Bemidji Super Cool School had 900 students:
That’s 464 sophomores,
  240 juniors and
  196 senior.
And they have 20 seats on the student council.

Divide the 900 students by the 20 seats to get 45 students represented by each seat. That is called the ideal ratio.

Divide the number of sophomores by the ideal ratio: \( \frac{464}{45} = 10.31 \)
Divide the number of juniors by the ideal ratio: \( \frac{240}{45} = 5.33 \)
Divide the number of seniors by the ideal ratio: \( \frac{196}{45} = 4.36 \)

If you award 10 seats to the sophomores,
  5 seats to the juniors, and
  4 seats to the seniors,
you have used 19 seats. How will you award the twentieth seat?

If you use the Hamilton method:
  Give the remaining seat to the class whose decimal part is largest, the seniors!
The sophomores win 10 seats, juniors 5 and seniors 5.

If you use the Jefferson method:
  Decrease your ideal ratio a little so that the quotas will rise a little. Adjust just enough that one of the classes passes the next integer mark, and that class gets the seat.
  Try this algorithm:
  1. Divide the total population by the number of seats to get an ideal ratio.
  2. Divide the population of each class (group) by the ideal ratio to get a class quota.
  3. Truncate the decimal points and award each class the integer number of seats.
  4. If the number of seats adds up to the total you want, you’re done!
  5. If the number of seats assigned is smaller than the total number of seats to give away, divide each class size by it’s (integer quota + 1). (The quotas will increase if the ideal ratio decreases.)
    This way you get an “adjusted ratio.”
  6. Award the seat to the class with the largest adjusted ratio. This works when you give away only one seat.
    The sophomores now have 11, the juniors 5 and the seniors, 4.
    Then you find a really good answer book to see how to handle a situation with two seats to give away.
If you use the **Webster method:**

Now this is a little different from what Todd said, so get a good book and check it out. When you divide each class size by the ideal ratio, you round the number up if the decimal portion starts with 5 or more; you round down if the decimal portion starts with a digit less than 5. If you then award the total number of seats, great. However, if you still have a seat to give away, you calculate “adjusted ratios” in a new way. If a class’s ideal ratio was 6.27, you would say it is between 6 and 7 and the arithmetic mean, 6.5 would be your adjusted ratio. Divide each class’s population by the new adjusted ratio and the class with the largest quotient gets the extra seat.

Now the sophomores have 11, juniors 5 and seniors 4.

Try the Hill Method:

When dividing by the ideal ratio doesn’t award all the seats, use yet another technique for an adjusted ratio: the geometric mean. It a class’s ideal ratio was 6.27, you would say it is between 6 and 7, so you would multiply 6 x 7 and take the square root of your answer. Then you would divide the class population by its adjusted ratio. The class with the highest adjusted ratio is awarded the seat.

Again, we get 11 for sophomores, 5 for junior and 4 for seniors.

Discuss which method(s) each class would like and dislike.
**Combination Problem**

Students at the Bemidji High School have a special name for Tuesdays, Taco Tuesday, because the school serves super-sized crunchy (hard shell) tacos filled with taco meat for lunch. (One taco will fill you up!) Students get to build on to their tacos by adding any of these ingredients: refried beans, cheese, sour cream, lettuce, tomatoes, black olives, and mild salsa.

How many different tacos are possible if students can only pick three of the seven additional ingredients to put on their taco?

How many different tacos can be made altogether using any number of additional ingredients?
Combination Problem using a Venn Diagram

The math club met during lunch last Tuesday and ate tacos during the meeting. Only 50 of the members attended. A vote was taken for three additional ingredients for the club members to choose from. (The principle would only allow three additional ingredients since the club was meeting in Mr. Frauenholtz’s classroom.)

The club voted for cheese, lettuce, and sour cream as the additional ingredients.
* 10 prefer only cheese  * 7 like cheese and lettuce
* 24 want sour cream  * 3 like cheese, lettuce, and sour cream
* 9 want lettuce  * no one wants only lettuce
* 7 want cheese and sour cream without lettuce

How many want a taco with only sour cream?

How many want a taco with sour cream and lettuce?

How many want a lettuce or sour cream taco?

How many want a plain taco (no additional toppings)?
**Venn Diagrams**

**Pilots**
- Some people got to take a class to become air traffic controllers.
- 35 were pilots.
- 20 were veterans.
- 30 pilots were not veterans.
- 50 people were neither pilots nor veterans.

How large was the whole group?

**Shakespeare**
- Really, the right way to do this is to have a sonnet in large print and have the students underline the verbs and circle the adjectives!

One of Shakespeare’s sonnets has 14 lines.
- 11 lines contain a verb.
- 9 lines have adjectives.
- 7 lines have both verbs and adjectives.

How many lines have a verb but no adjective?
How many lines have an adjective but no verb?
How many lines have neither an adjective nor a verb?

**This Venn problem introduces AND and OR.**

**Music**
- This problem should be modified for iPod contents!
- There was a survey of 190 students so learn what kind of music they preferred.
- 114 liked rock.
- 50 liked country.
- 41 liked classical.
- 14 liked rock and country.
- 15 liked rock and classical.
- 11 liked classical and country.
- 5 liked all three kinds of music.

How many liked rock only?
How many liked country but not rock?
How many liked classical AND country but not rock?
How many liked classical OR country but not rock?
How many liked exactly one style of music?
How many so not any of the three styles of music?
How many like at least two of the three styles of music?
How many do not like either rock or country?
Adjacent Circles

Connects “colored graphs” to combinatorics

From NCTM’s website, Illuminations lessons
Grades 6-8

Students will
Color a worksheet to discover how many ways there are to arrange three colors on a graph that has five vertices. (No adjacent vertices may be the same color.)

Because the vertices include a cycle (4 vertices form a connected rectangle), this exercise challenges students to create a tree diagram to match the colored patterns. The tree diagram has a simple but unique twist.

Comment: this is probably not a good activity when you have a sub. Poorly organized students will get off task. Could be used as a special activity at an “interest corner” or nice extra credit project.
**Sticks and Stones Game**

From NCTM’s website, Illuminations lessons

Students will:
- Collect data and graph frequencies
- Predict likelihood of various moves
- Use “expected value” to determine the average number of turns needed to win
- Modify rules to create a new game
- Think about theoretical vs. experimental probability

This is originally an Apache Indian game, so it is interestingly multicultural.

This is cool because students can lay out the game board in various ways. They start with 4 groups of ten steps around the board. By switching to 6 groups of 7 or to 3 groups of 12, they will apply modular arithmetic in their scoring! (A place marker is used so you do not rely on a numerical sum for a winning score.)

Materials required:
- Popsicle sticks
- Place markers (may use feathers or arrowheads for the theme.)
- About 40 small stones or chips to form a circle for the game board. (Suggestion: make a variety of game boards and glue the pieces down!)
**Exponential Bees**

Sorry, Jeff.

The number of African Killer Bees in Texas in 1987 was estimated to be 5,000.

It was also estimated that the bee population would increase about 12% per year.

So let the start of 1987 be “zero” as the beginning stage. Then let stage 1 represent the end of 1987, stage 2 as the end of 1988, etc.

Make a table of entries for the number of bees through the end of 1992.

Write a ‘now-and-next” equation.

Use a spreadsheet or calculator to find out when the bee population will be greater than 100,000.
Check That Digit

This lesson introduces students to a common and practical use of modular arithmetic. First the barcode system is examined, specifically UPC and ISBN bar coding. Then, students will discover the applications of modular arithmetic as applied to credit card numbers.

Learning Objectives

Students will:

- Determine the check digit for barcodes and credit card numbers.
- Test and confirm the validity of barcodes and credit card numbers using appropriate algorithms.
- Discover the importance of a check digit and explain its strengths and weaknesses.
- Compare and contrast different check-digit equations.

Materials

Check That Digit Activity Sheet
Credit cards from mail advertising or other sources
Examples of barcodes, ISBN, and UPC labels

Instructional Plan

Prior to the lesson, collect samples of barcodes, ISBN, UPC, and sample credit cards (from mail advertisements or from local financial institutions). Not only will these examples provide students with a visual, but they can also be used as part of the lesson when verifying the accuracy of the number. Begin by providing a background for the creation of barcoding systems and how these systems are used today.

The first patent for barcodes was issued to Bernard Silver and Norman Woodland in 1952. Since then these coding systems have expanded, been modified, and applied to a variety of areas. One of the most common uses is in retail and grocery stores. Although the barcode is not the price of the item, it does allow for the item to be registered with an associated price. When the bar code is scanned, the associated price will be read by the cash register. Other uses are found in monitoring blood supplies, identification on prescription drugs, book checkout at libraries, tracking luggage, and express shipping services.

Another advantage to the barcode system is that when an additional digit is included, known as the check digit, many errors that occur during data entry can be detected. These errors occur when passing information over the phone or internet. It is quite easy for people to transpose numbers (45 when it should be 54), replace a single digit with another, omit digits, or double an incorrect digit (799 is entered as 779). Using a check digit within a check equation helps to catch these errors and verify the validity of the number. As students will see in the ISBN barcode, the number is also used for identification purposes.

Begin by taking samples of barcodes that you have collected from items that are sold in stores.
Begin by taking samples of barcodes that you have collected from items that are sold in stores. This type of barcode is referred to as the Universal Product Code, or UPC. Students will be instructed as to the algorithm and then determine the validity of the number. Next, students should be challenged to determine a check digit for a barcode. The UPC system uses a mod 10 congruence. This system uses a weighting factor of 3 for the digits in the even positions. This means that even-positioned digits will be multiplied by three.

For the first example, use the given UPC symbol 7-86936-24425-0 from the movie "The Incredibles."

To verify this number, follow the steps:

1. Every even-positioned digit, counting from the right to left, will be multiplied by 3. All odd-positioned digits will be multiplied by 1.
   \[ 3(7) + 1(8) + 3(6) + 1(9) + 3(3) + 1(6) + 3(2) + 1(4) + 3(4) + 1(2) + 3(5) + 1(0) \]

2. Sum the products.
   \[ 110 \]

3. Determine the validity by dividing the sum by 10.
   \[ 110 \div 10 = 11 \text{ remainder } 0. \text{ Therefore } 110 \mod 10 = 0. \text{ This is a valid UPC number.} \]

Students may decide to apply the distributive property and multiply 3 by the sum of the even digits and multiply 1 by the sum of the odd digits.

Next provide students with the UPC number 7-96714-78601-\(y\), where \(y\) is the check digit. Using the process from above, students should determine the check digit. The sum of the products is 112. The check digit must be 8, because \((112+8) \mod 10 = 0\). On Question 1 of the Check That Digit activity sheet, students will verify the check digit for two UPC numbers.

Another barcode system is the International Standard Book Number, or ISBN. This system was developed in the late 1960’s and early 1970’s. It became apparent that there needed to be a uniform system that would identify books that were published throughout the world. Now every book could have a special identification number. The ISBN is a ten-digit number composed of
blocks of numbers that have different meaning. There are four parts to the number, which are separated by hyphens or spaces. The first part of the number identifies the language or country (referred to as the group identifier) and is at most five digits. The second part of the number identifies the publisher and may be at most seven digits. The third part of the number represents the item number or edition for that publisher. It may consist of no more than six digits. The final part is the check digit. Part of the flexibility of this system is the fact that there are many numbers available to be used. Recall that there are a maximum of 10 digits with the 10th being reserved for the check digit. Therefore, the first three parts of the number must have a combined total of nine digits. Leading zeroes are used as place fillers in the event there would not be enough digits in a particular section to ensure there are an appropriate number of digits. The diagram below shows an example of an ISBN number.

The check digit is calculated differently than that of the UPC system. Begin by multiplying the first digit by 10, the second by 9, the third by 8, and continue in this fashion until the ninth digit is multiplied by 2. Next, determine the sum of these products. This is a modulus-11 system, which means that the sum of the products of the first nine digits plus the check digit must be a multiple of 11. One problem that arises in this process is that the check digit might need to be a 10. Because we only have digits 0-9, an X is written in the check-digit place. (The X is reflective of the Roman numeral for 10.) Questions 3-6 on the Check That Digit activity sheet deal specifically with the ISBN. Note: Beginning January 1, 2007, the current ISBN system will be replaced with the ISBN-13 system (for further information, see the ISO Web Site). It is a 13-digit number beginning with 978, followed by the current nine digits of the ISBN, and it will have a new check digit. The check digit will be found using a method different from the current one. When all old ISBN’s have been used, the next series will begin with 979.

Credit cards use a system of blocked numbers similar to the ISBN. One obvious difference is that the maximum length for the number is 19 digits, although many numbers range from 13-16 digits.
The first digit of a credit card number is the Major Industry Identifier (MII) and identifies which group issued the card, as shown below.

<table>
<thead>
<tr>
<th>ISSUER</th>
<th>IDENTIFIER</th>
<th>CARD NUMBER LENGTH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diner’s Club/ Carte Blanche</td>
<td>300xxx -- 305xxx, 36xxxx, 38xxxx</td>
<td>14</td>
</tr>
<tr>
<td>American Express</td>
<td>34xxxx, 37xxxx</td>
<td>15</td>
</tr>
<tr>
<td>VISA</td>
<td>4xxxxx</td>
<td>13,16</td>
</tr>
<tr>
<td>MasterCard</td>
<td>51xxxx -- 55xxxx</td>
<td>16</td>
</tr>
<tr>
<td>Discover</td>
<td>6011xx</td>
<td>16</td>
</tr>
</tbody>
</table>

For instance, a number beginning with a 3 would be representative of the travel and entertainment category. The American Express card falls into this category. Cards issued by gas companies are given the beginning digit 7. The popular Visa and MasterCard fall under the banking and financial category (4, 5). The next block of numbers is the Issue Identifier. Including the MII digit, the Issue Identifier is six digits long. The account number begins with
the seventh digit and ends with the next-to-last digit. The final digit is the check digit.

![Credit Card Image]

The process used to determine the check digit is the Luhn algorithm (mod 10), named after IBM scientist Hans Peter Luhn. This algorithm works as follows:

1. Begin by doubling all even-positioned digits when counting from right to left.
2. Determine the sum of the digits from the products (Step 1) and each of the unaffected (odd-positioned) digits in the original number.
3. Verify the account number by determining if the sum from step 2 is a multiple of 10.

Before proceeding to the questions on the activity sheet pertaining to this topic, have students become more familiar with the Luhn algorithm by determining the validity of the check digit for the following account number: 5314772685932112. The sum produced by the algorithm is 101, found as follows:

\[2(5) + 3 + 2(1) + 4 + 2(7) + 7 + 2(2) + 6 + 2(8) + 5 + 2(9) + 3 + 2(2) + 1 + 2(1) + 2 = 101\]

To be a valid account number, this sum must be evenly divisible by 10. If the check digit were 1, the result would be congruent to 0 mod 10; but because the check digit is 2, the sum is not divisible by 10. Therefore, this account number is not valid.

The Luhn algorithm is able to detect single data entry errors and most transpositions. Students should proceed to the worksheet and determine how this happens.

Prior to beginning the lesson, you may wish to review the solutions.

Questions for Students

A problem of the UPC system is that if two adjacent digits that were transposed have a difference of 5, the error will not be detected. Explain why this occurs. [When the original digits are multiplied by 1 and 3 and the transposed digits are multiplied by 1 and 3, the difference of the two sums is 10. This is a problem because the sums of both UPC numbers will yield a remainder of zero when divided by 10.]

As we have seen on many television commercials, there are many banking institutions that offer credit cards. The first six digits that appear on a credit card are used for the issue
identifier. How many possible issuers are there given each digit 0-9 could be used more than once?

[There would be 106 or 1,000,000 possible issue identifiers.]

This process is also used to detect most digit transpositions. For instance when entering a number 5832403 the data entry error is transposing the second and third digits: 5382403. There are two digits, when transposed that will go undetected using the Luhn algorithm. What are they? Explain why this error cannot be detected.

[The digits that cannot be detected are 0 and 9. These are unique because the value of these two digits will always be a 0 and 9 regardless of their position in the account number. If the 9 is in the even-numbered position it will be doubled resulting in 18 with a sum of 1+8=9 and the 0 in the odd-numbered position would be unaffected. On the other hand, if 0 were in the even-numbered position, its value doubled would still be 0 and when added to the 9 the sum is still 9. Either way the sum of the two is nine.]

확정된 텍스트입니다.

Assessment Options

1. One option is to have the students complete the Activity Sheet for homework. Then over the next week, ask students to find examples of credit card numbers that they see in advertisements or through the mail and verify if the number is valid. It might be a good idea to remind students that unauthorized use of credit card is a form of fraud and is punishable by law. Credit card numbers, like social security numbers, should be protected by their owners to limit identity theft.

2. Another option is to have students develop another type of check equation for an imaginary credit card company. The equation should use a similar system to those discussed in class, but vary the weighting factor, the modulus, and the number of digits.

3. To involve the community, students could interview a professional at a local banking institution and write a short report based on the interview. The report should focus on how the account numbers are coded, how that information is sent/transferred, and what security measures are in place to prevent account number theft.

확정된 텍스트입니다.

Extensions

1. Students may research the changing ISBN system from the current 10-digit format to the new 13-digit format. Students should explain the reasons for the new system and determine the benefits and any problems that may occur. They should also determine what types of entities are affected by this major change: businesses, schools, libraries, publishing companies, etc.

2. Students can research other types of codes. Codes are found in the following places:
   - library patron’s library cards
   - UPS and Fed Ex to track the shipping of packages
   - zip codes used by the US Postal service
   - supermarket club cards

Students should determine the check digit scheme that is used and if the numbers in the code have any specific categorization use.
Teacher Reflection

- How did the students demonstrate understanding of the materials presented?
- This lesson shows the importance mathematics plays in something that millions of people use each day. Did the students gain an appreciation for mathematics? Were they interested in learning about how math is used in part of everyday life?
- What other lessons could be developed that would demonstrate the practicality of math?
- What grouping approach did you choose for this lesson? Partners, groups of 3 or 4? Was this approach effective? Why or why not? What would you change for next time?
- Were concepts presented too abstractly? Too concretely? How would you change them?
- Did you find it necessary to make adjustments while teaching the lesson? If so, what adjustments, and were these adjustments effective?

NCTM Standards and Expectations

Number & Operations 9-12

1. Use number-theory arguments to justify relationships involving whole numbers.
2. Develop an understanding of permutations and combinations as counting techniques.
3. Develop fluency in operations with real numbers, vectors, and matrices, using mental computation or paper-and-pencil calculations for simple cases and technology for more-complicated cases.
4. Judge the reasonableness of numerical computations and their results.

This lesson prepared by Doug Schmid.
Check That Digit NAME ___________________________

How does a register at the supermarket accurately scan item numbers? Can mistakes in scanning be detected? The purpose of this activity is to see how modular arithmetic is applied to UPC and ISBN bar coding. Credit card numbers also use modular arithmetic to verify card numbers.

1. Verify the check digit for each of the two UPC numbers by doing the following steps:
   a) Every even-positioned digit, counting from the right to left, will be multiplied by 3. All odd-positioned digits will be multiplied by 1.
   b) Sum the products from step a.
   c) Determine the validity by dividing the sum by 10. If the remainder is 0, the UPC number is valid.

   a) 0-87684-00974-3
   b) 0-43197-11682-6

2. A problem of the UPC system is that if two adjacent digits that were transposed have a difference of 5, the error will not be detected. Explain why this occurs.

3. In general, larger publishing companies have a small identification number (the second block of the ISBN) and the smaller companies have a larger number. Explain why this is true.

4. For a publishing company that has 81 as its publisher identifier and 1 for its language/country identifier, determine the number of possible editions this publisher may print. The digits may be repeated. (Note: The ISBN number is 9 digits plus a check digit.)

5. What if the digits were not repeated. How would this affect the number of published editions? What would be the total possible?

6. The ISBN system is better at detecting errors, specifically transposition errors. Question 2 addressed this problem with the UPC system. Explain why, unlike the UPC system, the ISBN system will detect all transposition errors.
7. As we have seen on many television commercials, there are a lot of banking institutions that offer credit cards. The first six digits that appear on a credit card are used for the issue identifier. How many possible issuers are there given each digit 0-9 could be used more than once?

8. Given that a credit number can have as many as 19 digits, six of which are reserved for the issue identifier and the last is the check digit, determine the total number of credit card numbers that are available to each issuer.

9. MasterCard issues identifier numbers are 6 digits in length and begin with either 51 or 55. How many possible issue identifier numbers are there for MasterCard?

10. Using the Luhn algorithm, determine the check digit for an account number of 601143871005123__

The Luhn algorithm is as follows:

a) Begin by doubling all even-positioned digits when counting from right to left.

b) Determine the sum of the digits from the products (step a) and each of the unaffected (odd positioned digits) digits in the original number.

c) Determine the number to be added to the sum from step b so that this new sum is a is a multiple of 10. This number is the check digit.

11. In the above process there are two sums you are finding: one involving the odd-positioned digits and the other involving the even-positioned digits that were doubled. First consider the sum of the odd-positioned digits. What is the most the sum could be affected by if an incorrect digit was entered? Explain why this occurs.

12. Next consider the even-positioned digits that were doubled. It is possible here to have numbers that result in double digits. Recall how this sum is determined. How is it possible that a single digit error would be detected? [Hint: Make a table which shows possible values for the check digit, the double of the check digit and the resulting sums.]

13. The Luhn algorithm is also used to detect most digit transpositions. For instance when entering a number 5832403 the data entry error is transposing the second and third digits: 5382403. There are two digits, when transposed that will go undetected using the Luhn algorithm. What are they? Explain why this error cannot be detected.
14. Are there any other circumstances under which a transposition error would go undetected? If so, would this cause a problem?

15. What other types of accounts or cards would you expect may use a process similar to the Luhn Algorithm?
DEEP IN THE HEART OF SOUTHERN Louisiana's Cajun country, middle school students enter a mathematics classroom. They hear Cajun music playing in the background and observe cattails and cypress knees "growing" along the bottom of the classroom walls. Foam creole fish lie on plastic bayous on the tables around the room. The room is filled with excitement for what is about to occur. These students begin to question each other, and ask: "Is this math?" "Perhaps we're going to skip math and do social studies?" "Maybe we are going to learn how to do a Cajun line dance?" They quickly try to make sense of this atypical mathematics classroom.

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The authors wish to extend their deepest gratitude to Joy Jelks for her help in making this article possible.
environment. Through a culturally relevant context, these students are about to experience a branch of discrete mathematics called graph theory. Specifically, they will learn about odd and even vertices of finite graphs as well as Euler (pronounced “oil-er”) circuits and paths. Mathematical connections to their real world will be made evident using small-group and whole-class discussions throughout the lesson. According to the Connections Standard in Principles and Standards for School Mathematics, all students should be able to “recognize and apply mathematics in contexts outside of mathematics” (NCTM 2000, p. 65).

**Using Discrete Mathematics to Promote Connections**

**ALL TOO OFTEN, STUDENTS LEARN TO VIEW mathematics as a collection of isolated topics rather than as a coherent body of concepts. For students to begin to visualize mathematics as a coherent body of concepts, they need to engage in activities that make connections between mathematics and their lives (Kennedy, Tipps, and Johnson 2004). According to Monroe and Mikovich, “Ample evidence is available to support the contention that, for learning to be meaningful, concepts must be connected and integrated within the experiences of the learner” (1984, p. 371). By posing problems in contexts that are relevant to students, connections are naturally made between the mathematics needed to solve the problems and the context in which the problem is posed. When the context is familiar to students and the problem is easily understood, students may not realize that they are actually using mathematics to solve the problem. Such is the case for most problems that arise in discrete mathematics.**

Discrete mathematics is the branch of mathematics that involves the study of objects and ideas that can be divided into ‘separate’ or ‘discontinuous’ parts” (Dossey 1991, p. 1). In discrete mathematics, functions defined for a finite set of numbers, such as a finite set of positive integers, can be studied. Thus, when students are solving problems whose answers can only be whole numbers, they are solving discrete mathematics problems (e.g., involving existence, “yes/no” problems; counting; or optimization, problems with multiple solutions, but only one of which is “best”) (Dossey 1991). Discrete mathematics is easily applied to contexts outside of school mathematics, and its study can motivate students.

Because discrete mathematics lends itself well to real-world applications, it is easily incorporated into culturally relevant mathematics lessons, referring to lessons experienced within a cultural environment (Hatfield, Edwards, Bitter, and Morrow 2005). Students should be aware of the mathematics present in their own cultures so that they can begin to develop an appreciation and an understanding for how mathematics is used in their own real worlds. According to Zaslavsky (1996), the mathematics presented to students through a cultural context must be realistic and of interest to the students. Otherwise, it is not truly culturally relevant. By engaging in such culturally relevant lessons, students learn to respect and value their own culture and the cultures of others (Hatfield, Edwards, Bitter, and Morrow 2005).

**The Lesson: Let’s Go Crawfishing**

**THE LESSON CAN BE TAUGHT EITHER IN ONE ninety-minute period or over two consecutive one-hour periods. During the lesson, students will engage in a hands-on activity designed to guide them toward discovering when a graph contains an Euler circuit or an Euler path.** The following materials are needed for the lesson:

- Ten separate graphs, representing mazes of south Louisiana bayous
- Black electrical tape
- Clear plastic polypropylene sheeting
- Manipulative foam shapes (crawfish, in this case)
- Velcro (see **fig. 1** for an example using the sheeting, electrical tape, and foam shapes)
- Recording sheets (see **fig. 2**)
- Sketch sheets (see **fig. 3**)
- Transparencies of each of the ten graphs
- Transparency pens
- Overhead projector
- Appropriate cultural décor for the classroom (i.e., swamp scenes including cypress knees, cattails, crawfish, and bayous)

![Fig. 1 Sample bayou maze stations](image-url)
As students enter the classroom, they will see that information is posted on the walls about the lesson that is about to occur. This information includes the goal of the lesson, which is to determine whether or not all the crawfish on a graph can be “caught” without tracing over parts of the graph twice. In mathematical terms, the goal of the lesson is to determine if a graph has an Euler path or Euler circuit and to be able to provide a rationale as to why this is the case. Because this lesson uses cooperative groups of four, the individual jobs of the group members are also posted on a walk recorder, sketcher, manipulator, and timekeeper/noise controller. The recorder is responsible for completing the “Bayou Crawfishing Recording Sheet” (fig. 2). The sketcher records how the group caught the crawfish on each maze using the “Bayou Crawfishing Sketch Sheet” (fig. 3). The manipulator physically removes the crawfish from each graph as the group comes to a consensus as to the order that the crawfish should be removed, then puts all crawfish back in their original positions before the group moves on to the next graph. Finally, the timekeeper/noise controller keeps track of the start time at each crawfish graph station, informs the group when one minute remains, and monitors the group’s noise level. A minimum of five minutes is given for each station, but additional time should be available if students need it.

The teacher begins the lesson by explaining that students are going to catch crawfish and that the lesson is divided into four parts. The students are to work in cooperative groups and complete the following four steps:

1. Determine if there is a way to catch all the crawfish on each bayou maze without going over any bayou twice.
2. If this is impossible, figure out how to catch as many crawfish as possible without going over any bayou twice, and record how many crawfish were not caught.
3. If it is possible to catch all the crawfish, then determine if it is possible by starting and ending at the same point (vertex) on the bayou maze.
4. Be prepared to share your findings with the class and explain the results.

After describing the task, the teacher gives the students some background information on the mathematical terms that they will be using. A graph is a collection of points, called vertices. Some or all of these vertices are

<table>
<thead>
<tr>
<th>Graph No.</th>
<th>No. of Vertices</th>
<th>No. of Even Vertices</th>
<th>No. of Odd Vertices</th>
<th>Is It Possible to Catch All Crawfish?</th>
<th>Is It Possible to Start and End at the Same Spot?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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</tbody>
</table>

What do you notice about the data in the table above?

Make a conjecture about when it is possible to catch all the crawfish and when it is not possible.

Make a conjecture about when it is possible to start and end at the same spot, having caught all crawfish.

Fig. 2 Students fill out this sheet to start the activity.
joined by line segments or curves, called edges. In the context of this lesson, all edges connect two distinct vertices and all vertices are connected to one another by one or more edges. It is also explained that a vertex, signified as a big black dot where two bayous meet, is classified as either even or odd, depending on the number of edges that go into it (see fig. 4). The teacher then explains each of the four group jobs, instructing the students on how to fill in the “Bayou Crawfish Fishing Recording Sheet” (fig. 2) and the “Bayou Crawfish Sketch Sheet” (fig. 3), and points out the location of the ten Bayou Crawfishing stations, each of which has one maze. Each group is assigned a station and will rotate from station to station. Some stations may not have a group assigned to them at the start of the activity.

The cooperative groups then determine if all the crawfish on each bayou maze can be caught without going over any bayou twice. While the students are working on each maze, the teacher monitors each group by observing the answers on the recording sheets, listening to students’ rationales for why it can or cannot be done, and providing students with hints or prompts as needed. After students have investigated four or five of the bayou mazes, prompts may be used, such as these:

- Have you tried starting at both an even and an odd vertex?
- When approaching a certain vertex, what must you be sure of? Why is this important? How does this relate to the vertex being even or odd?
- Is it possible to start and end at every vertex, or is it only possible to start and end at certain vertices?
- If you started on an even vertex, did you end on an even vertex? If so, was it the same even vertex or a different one?
- If you started on an odd vertex, did you end on an odd vertex? If so, was it the same odd vertex or was it a different one?

Once all groups have completed the recording and sketch sheets, the teacher and students discuss the answers and possible ways to catch the crawfish (see figs. 5 and 6). Although the answer for each graph is either yes or no, when students indicate yes, the teacher should solicit several ways in which all crawfish could be caught. Students become aware that there are different ways to catch all the crawfish, thus the path taken is not unique. When the students indicate a no answer, they need to explain all the different ways to get as many crawfish as possible. Once the students agree on the correct answers, the teacher guides students to discover when it is both possible and impossible to catch all crawfish without going over a bayou twice. Using the previously stated prompts, the teacher provides
Completed Bayou Crawfishing Recording Sheet

<table>
<thead>
<tr>
<th>GRAPH NO.</th>
<th>NO. OF VERTICES</th>
<th>NO. OF EVEN VERTICES</th>
<th>NO. OF ODD VERTICES</th>
<th>IS IT POSSIBLE TO CATCH ALL CRAWFISH?</th>
<th>IS IT POSSIBLE TO START AND END AT THE SAME SPOT?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>8</td>
<td>8</td>
<td>0</td>
<td>yes</td>
<td>yes</td>
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<tr>
<td>2</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>yes</td>
<td>no</td>
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<td>2</td>
<td>4</td>
<td>no</td>
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<td>7</td>
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<td>2</td>
<td>yes</td>
<td>no</td>
</tr>
</tbody>
</table>

What do you notice about the data in the table above?

*Answers will vary.*

Make a conjecture about when it is possible to catch all the crawfish and when it is not possible.

*Answers will vary.*

Make a conjecture about when it is possible to start and end at the same spot, having caught all crawfish.

*Answers will vary.*

*Fig. 5 A completed recording sheet helps students see possible patterns.*

Hints concerning the role of even and odd vertices and starting and ending vertices.

Using the data collected from the ten graphs and teacher prompts, the students should realize that it is possible to catch all the crawfish as long as the graph contains no more than two odd vertices. They can then make a conjecture regarding when it is possible to catch all the crawfish by starting and ending at the same vertex. Students are then asked why it would be beneficial to start and end at the same vertex, such as for economic and logistical reasons involving fuel costs and time issues. After the students discover that it is possible to start and end at the same vertex when all vertices are even, the teacher then introduces the mathematical terms *Euler path* and *Euler circuit* for the two possibilities of successfully catching all the crawfish.

The lesson concludes with explanations of an Euler path and an Euler circuit. An Euler path refers to a path in a graph that uses each edge exactly once; vertices may be repeated, but not edges. Thus, in the context of this lesson, if it was possible to catch all the crawfish without going over a bayou more than once, then that graph contained an Euler path. Additionally, an Euler circuit is an Euler path that ends where it begins. Therefore, if it was possible to catch all the crawfish without going over a bayou more than once and start and end at the same vertex, then that graph contained an Euler circuit. The teacher should also prompt the students to notice that any vertex can be the starting and ending vertex if the graph contains all even vertices. However, when a graph contains two odd vertices, one odd vertex must be the starting point and the other must be the ending point. Additionally, students need to realize that an Euler path or an Euler circuit may not be unique. After some discussion, the class should categorize the ten bayou mazes as containing an Euler path and/or an Euler circuit. Finally, for those bayou mazes that contain neither one, the teacher should discuss why it is important to catch as many crawfish as possible; e.g., the more crawfish caught, the more money earned or the more food to eat.
Completed Bayou Crawfishing Sketch Sheet—One Possible Solution

Sketch your solution to each of the mazes.

1. Place an “S” next to the vertex that you started on.
2. Draw arrows on the bayous to denote the direction that you went to get each crawfish.
3. Place an “E” next to the vertex that you ended on.

Two copies of each maze are provided—one to show how all crawfish were caught and one to show how this was done by starting and stopping at the same spot.

Fig. 6 A completed sketch sheet shows students’ paths.
As a follow-up to this lesson, students could design their own bayou mazes (graphs) that contain Euler paths and Euler circuits, then verify their previous conjectures by using arrows showing the Euler path or Euler circuit. They could also construct and investigate if it is possible to construct a maze (graph) with only one odd vertex to verify that it does not have an Euler path (a graph cannot have only one odd vertex). Additionally, students could create other scenarios in which Euler paths and Euler circuits could occur, such as when delivering mail or feeding animals in a zoo. Presenting a similar problem but in a different context will determine if students realize that it is really the same problem and that the solution is evident. For example, in keeping with the culture of southern Louisiana, students could be asked if certain Mardi Gras masks could be traced or drawn without lifting their pencils (see Fig. 7).

Conclusion

Teachers who have used this culturally relevant lesson have commented about how it motivated their students; how problem-solving skills were improved; and how a conceptual understanding of Euler paths, Euler circuits, and even and odd vertices occurred by participating in the lesson. One teacher remarked, “Even the students who usually have difficulty with math are able to engage in this lesson.” Additionally, students commented on how “fun these puzzles are,” indicating that they do not view this activity as a real mathematics lesson but rather as some kind of challenging enrichment task. The element of surprise and the simplicity of the lesson are natural motivators.

Understanding the mathematics concepts being taught is evident during the discussion of when it is possible to catch all the crawfish by starting and stopping at the same vertex. After studying the data collected on all ten bayou mazes, students who have participated in this lesson display insight as they pose conjectures and make observations such as these:

- “There is always an even number of odd vertices.”
- “When there are no odd vertices, you can start and end at the same spot.”
- “It doesn’t work for more than two odd vertices.”

This culturally relevant lesson can be easily adapted to other regions of the United States. Any crop that is grown in a particular area could be harvested without going over any path twice, since the crop was already picked. For example, peaches could be harvested in Georgia, potatoes in Idaho, or grapes in California. Similarly, cattle in Texas, Nebraska, or Iowa could be fed along maze paths that can be traveled no more than once. If none of these contexts is culturally relevant, generic scenarios can include paths or routes for postal delivery, airplane travel, school bus travel, or animal feeding at the local zoo. Incorporating all the NCTM’s Process Standards—Reasoning and Proof, Problem Solving, Communication, Connections, and Representation—makes this lesson worthy of consideration for any middle school mathematics classroom.

References


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