Compound and Chaotic Motion in the Double Pendulum System

David P. Gurney
Senior, Physics/Honors Program
Bemidji State University

This thesis explores the behavior of the double pendulum system using the computing power of Mathematica. Specific attention is paid to the system's sensitivity to initial conditions, a characteristic of chaos. The double pendulum system is a good system through which to study chaotic motion because it displays both compound and chaotic motion, depending on the amount of energy present in the system. Poincare sections are determined for a variety of initial energies, allowing for visual representation of the regions of compound and chaotic behavior.

Faculty sponsor: Dr. David A. Bahr

Introduction

The double pendulum system consists of two simple pendula, where the second pendulum is attached to the end of the first, with each pendulum length having a mass attached to the end. For the purposes of this thesis the pendulum lengths will be considered as massless, so that the entirety of systems mass resides in the two masses contained on the end of each pendulum length. Frictional forces are also ignored for purposes of simplification. The system is acted upon by the gravitational acceleration g. A diagram of the double pendulum system is provided below:

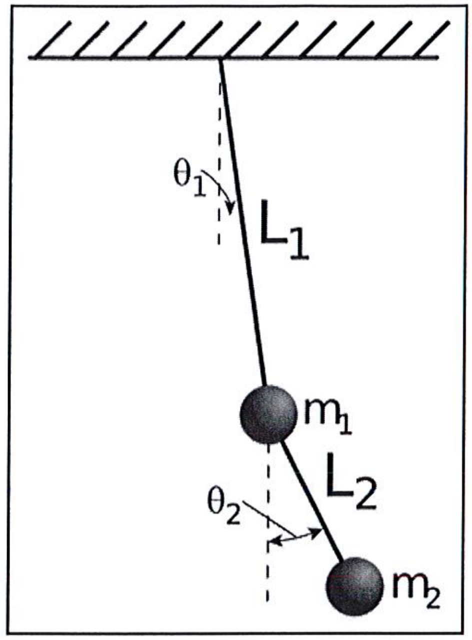

Figure 1: Diagram of a double pendulum system

There are multiplicities of variations of the double pendulum system. This thesis deals only with one of these varieties. In addition to the simplifications noted above, the double pendulum system studied in this thesis is un-driven and un-damped. Also, of the six parameters that can be seen in the figure above, this thesis deals exclusively with variations in θ_1 and θ_2.

The double pendulum system exhibits rich dynamical behavior. This makes it a perfect candidate for the study of chaotic motion, as it provides the quintessential example of a simple physical system that can display surprisingly complex motion under certain conditions.[1]

Lagrangian Mechanics

Isaac Newton described the first formulation of classical mechanics in the 17th century. Newton explained that the motion of a system could be described by three parameters: position, mass, and force.[4] When describing the motion of any system one must solve for the time-varying forces involved.[15] This approach can become cumbersome when dealing with systems such as the double pendulum. Even to describe the motion of a single pendulum system requires solving a large number of equations that account for the forces that work to maintain the path of the pendulum. Joseph-Louis Lagrange developed a simpler, and significantly more general approach in 1788.[8] Lagrange replaced Newton's rather tenuous notion of forces with the more intuitive and rigorous notions of energy and momentum. Lagrange also introduced generalized coordinates (discussed in greater detail below) to classical mechanics-a method that makes the analysis of a system's motion much simpler and more intuitive.[8] Although the derivation and manipulation of Lagrange's mechanics may require a greater degree of mathematical abstraction, the analysis of a system's motion becomes significantly more generalized under Lagrange's system.[15]

Generalized coordinates

The analysis of a system's behavior in classical mechanics is greatly simplified through the use of generalized coordinates.[3] Rather than rely on a system of coordinates whose descriptions of motion are dependent on one another, generalized coordinates allow us to choose a system of coordinates that are fundamentally independent and still allow for a complete description of the system's motion.[3] For the double pendulum, without the use of generalized coordinates we would be required to describe the interdependence of the x and y coordinates of the system and the accompanying time-varying forces. By choosing a set of appropriate generalized coordinates we can eliminate theses forces from our calculations of the system's motion. It is convenient for the double pendulum system to choose our generalized coordinates as θ_1 and θ_2, the two independent parameters that most simply describe the motion of the system at any given time. Via this choice of coordinates we have reduced the description of a system that in Cartesian coordinates would require four coordinates (x_1, y_1, x_2, y_2) to only two (θ_1, θ_2), which matches the system's number of degrees of freedom. The transformation of the Cartesian coordinate system onto the generalized coordinate system is as follows:

$$(x_1, y_1) = (l_1 sin\theta_1, l_1 cos\theta_1)$$

$$(x_2, y_2) = (l_1 sin\theta_1 + l_2 sin\theta_2, l_1 cos\theta_1 + l_2 cos\theta_2)$$

Euler-Lagrange equations

In order to derive the Euler-Lagrange equations necessary for our study of the double pendulum system we must begin with a description of the calculus of variations. A general principle of the calculus of variations is the determination of extremum solutions, for example, the shortest distance or time between two points.[15] To begin, we wish to determine the function $y(x)$ such that the integral:

$$J = \int_{x_1}^{x_2} f\left\{y(x), y'(x); x\right\} dx$$

is an extremum.[15] We then vary the function $y(x)$ until an extremum value of J is found. This means that if a function $y = y(x)$ gives the J a minimum value, then any neighboring function, no matter how close to $y(x)$, must make J increase. We define a neighboring function as:

$$y(\alpha, x) = y(0, x) + \alpha\eta(x)$$

where we give all possible functions y a parametric representation $y = y(\alpha,x)$ such that, for $\alpha = 0$, $y = (0, x) = y(x)$ is the function that yields an extremum for J.[15] $\eta(x)$ is some function of x that has a continuous first derivative and vanishes at x_1 and x_2. This is because the varied function $y(\alpha, x)$ must be identical with $y(x)$ at the endpoints of the path: $\eta(x_1) = \eta(x_2) = 0$. This explanation of extrema and neighboring function is most easily understood visually:

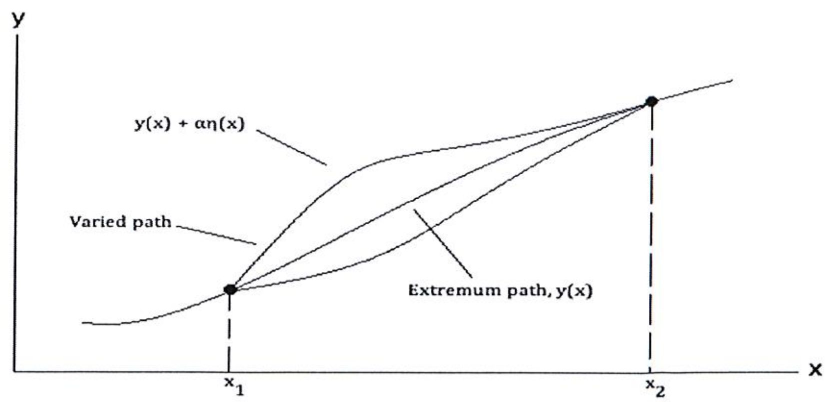

***Figure 2*: The function *y*(*x*) is the path that makes the function *J* an extremum. The neighboring functions *y*(*x*) + $\alpha\eta(x)$ vanish at the endpoints and may be close to *y*(*x*), but are not the extremum.**[15]

It follows that the integral J becomes a functional, which is a function that takes a vector as its argument or input and returns a scalar, of the parameter α:

$$J(\alpha) = \int_{x_1}^{x_2} f\left\{y(\alpha,x), y'(\alpha,x); x\right\} dx$$

The condition that the integral must have an extremum is that *J* be independent of α in first order along the path yielding the extremum ($\alpha = 0$):

$$\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$$

for all functions $\eta(x)$. Therefore:

$$\frac{\partial J}{\partial \alpha} = \frac{\partial}{\partial \alpha}\int_{x_1}^{x_2} f\left\{y, y'; x\right\} dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\frac{\partial f}{\partial \alpha} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \alpha}\right) dx$$

And

$$\frac{\partial y}{\partial \alpha} = \eta(x); \quad \frac{\partial y'}{\partial \alpha} = \frac{\partial \eta}{\partial x}$$

So, we have:

$$\frac{\partial J}{\partial \alpha} = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y}\eta(x) + \frac{\partial f}{\partial y'}\frac{d\eta}{dx} \right) dx = \int_{x_1}^{x_2} \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} \right) \eta(x)\, dx$$

Since $\eta(x_1) = \eta(x_2) = 0$. Although the integral appears to be independent of α, the functions y and y' with respect to which the derivatives of f are taken are still functions of α. But $\left.\frac{\partial J}{\partial \alpha}\right|_{\alpha=0} = 0$ for the extremum value and $\eta(x)$ is an arbitrary function, so the integrand must itself vanish for $\alpha = 0$.[9, 15] We now obtain the Euler-Lagrange equation where y and y' are the original functions, independent of α:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

Chaos

The irregular and unpredictable time evolution of many nonlinear systems has been dubbed 'chaos'.[3] However, a true definition of chaos has yet to be agreed upon by the scientific community.[5] In addition to the above definition, perhaps the most common description of chaos is a system that shows extreme sensitivity to initial conditions.[7] For instance, weather systems are considered to exhibit chaotic behavior due to small variations in a local weather system having the potential to cause major changes in weather across the globe. This is often termed 'the butterfly effect', a reference to the minute amounts of wind generated by the movement of a butterfly's wings that could lead to the development of some drastic weather pattern, such as a hurricane, on the other side of the globe.[5] In terms of pendulum systems, chaos only becomes apparent for large energies of the system. In other words, if the energy is small enough that we may use the small-angle approximation to determine the systems motion, we can be sure that chaos will not present itself. However, if the energy is increased beyond this point, complex behavior in the system becomes common, and for certain energies chaos may develop.[3] What is so interesting about chaos is its unpredictable nature. For instance, if a pendulum exhibited simple behavior for an initial energy of .3, complex behavior for energies .4, .5, then for an energy of .6 the system may suddenly become chaotic only to return to simple behavior for an initial energy of .7. This situation becomes even more pronounced in a more complex system such as the double pendulum[18], where motion is often unpredictable, as can often be seen with the naked eye. Determining the initial conditions that lead to chaos within a system can be very important. Consider an amusement park ride modeled on the double pendulum system. It would be extremely important for the designers and operators of the ride to be aware of what initial energies lead to chaotic behavior so that they may either be sure to avoid these initial conditions, or implement safety systems that take the potential of deterministic chaotic motion into account.

Poincare sections

Henri Poincare (1854-1912) invented a technique to help visualize the behavior of dynamical system's.[13] Poincare's method was to take a stroboscopic view of a dynamical systems 3*D* phase diagram and intersecting the resulting diagram at equal intervals with parallel planes.[15] The points at which the systems function intersects the plane are plotted as points in 2*D* space.[14] The number of points plotted will then correspond to the period of the system. This is enormously helpful in determining when simple, complex, or chaotic behavior is present in the system.[3] A Poincare section displaying a single point is simple because for each period the system returns to the same position. If the Poincare section shows many points we know that the system is complex because it does not return to the same position after each period.[15] As the number of points on the Poincare section increases the system is displaying ever more complicated behavior. As the period of the system becomes ever larger, the number of points on the Poincare section may reach a point where periodic behavior is no longer apparent. We would call such cases where the period approaches infinity chaotic. Poincare sections are especially helpful under the circumstances of chaos.[3] With a traditional phase portrait projected onto 2*D* space, a period approaching infinity will create a display that is essential 'full' of points so that no structure whatsoever can be discerned. Poincare sections eliminate this problem by taking mapping only the points at which the individual phase trajectories return to the same position. Poincare sections are only helpful for systems that obey Liouville's theorem, which states that if a system preserves volume and has only bounded orbits, then for each open set there exist orbits that intersect the set infinitely often.[1] Poincare sections also assume the following two stipulations on the system in question: phase trajectories do not intersect in closed dynamical systems, and the phase volume of a finite element under dynamics is conserved.[1] For all of the complicated language, Poincare sections are actually fairly easy to understand when presented visually. Below is a diagram representing compound motion within a dynamical system. On the left is a stroboscopic view of the phase portrait for the system, and on the right is the Poincare section derived from the diagram on the left. The Poincare section takes only select points from the phase diagram at an interval determined from the period of the function in question. This reduces the complexity of the diagram on the left to the more simple representation on the right.

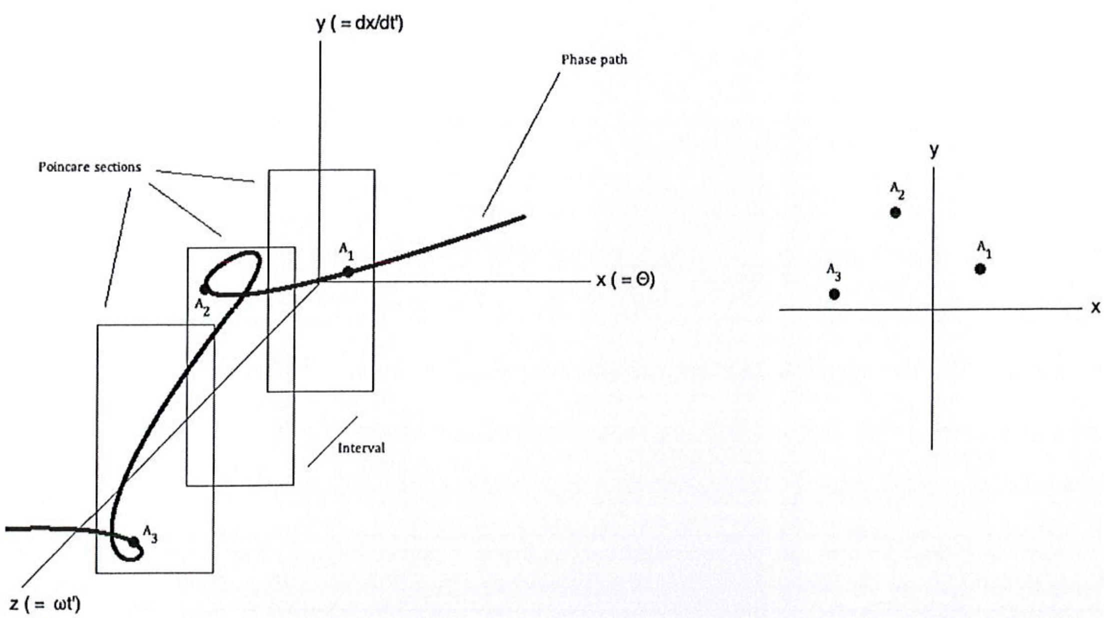

Figure 3**: Left-hand side – stroboscopic view of phase diagram. Right-hand side – associated Poincare section.**[15]

Formalism

We define the double pendulum system from *Figure 1* as follows:

x = horizontal position of pendulum mass

y = vertical position of pendulum mass

θ = angle of pendulum

l = length of pendulum

m = mass of pendulum

We take the pendulum lengths to be without mass, so that the entire mass of each pendulum is contained within a point-particle that we call m.

We now take the generalized coordinates to be θ_1 and θ_2:

$$
\begin{aligned}
x_1 &= l_1 \cos\theta_1 \\
y_1 &= -l_1 \cos\theta_1 \\
x_2 &= x_1 + l_2 \cos\theta_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 \\
y_2 &= y_1 - l_2 \cos\theta_2 = -l_1 \cos\theta_1 - l_2 \cos\theta_2
\end{aligned}
$$

And the energy of the systems is as follows:

Potential energy:

$$V = mgh = mgy_1 + mgy_2 = -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 g l_2 \cos\theta_2$$

Kinetic energy:[9]

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

So the Lagrangian is:[9, 18]

$$L = T - V = \frac{1}{2}\left(m_1 + m_2\right) l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{\theta}_1 \dot{\theta}_2 \cos\left(\theta_1 - \theta_2\right) + \left(m_1 + m_2\right) g l_1 \cos\theta_1 + m_2 g l_2 \cos\theta_2$$

From which the Euler-Lagrange equations follow:

For θ_1:

$$\frac{\partial L}{\partial \theta_1} = -\left(m_1 + m_2\right) g l_1 \sin\theta_1 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\left(\theta_1 - \theta_2\right)$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos\left(\theta_1 - \theta_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \left(m_1 + m_2\right) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos\left(\theta_1 - \theta_2\right) - m_2 l_1 l_2 \dot{\theta}_2 \sin\left(\theta_1 - \theta_2\right)\left(\dot{\theta}_1 - \dot{\theta}_2\right)$$

Therefore:

$$\frac{\partial L}{\partial \theta_1} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \left(m_1 + m_2\right) g \sin\theta_1 + \left(m_1 + m_2\right) l_1 \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 \cos\left(\theta_1 - \theta_2\right) + m_2 l_2 \dot{\theta}_2^2 \sin\left(\theta_1 - \theta_2\right) = 0$$

For θ_2:

$$\frac{\partial L}{\partial \theta_2} = m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\left(\theta_1 - \theta_2\right) - m_2 g l_2 \sin\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - m_2 l_1 l_2 \dot{\theta}_1 \cos\left(\theta_1 - \theta_2\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos\left(\theta_1 - \theta_2\right) - m_2 l_1 l_2 \dot{\theta}_1 \sin\left(\theta_1 - \theta_2\right)\left(\dot{\theta}_1 - \dot{\theta}_2\right)$$

Therefore:

$$\frac{\partial L}{\partial \theta_2} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 g \sin\theta_2 + m_2 l_1 \ddot{\theta}_2 + m_2 l_1 \ddot{\theta}_1 \cos\left(\theta_1 - \theta_2\right) - m_2 l_1 \dot{\theta}_1^2 \sin\left(\theta_1 - \theta_2\right) = 0$$

Results

A Mathematica program was developed by Dr. David Bahr and I to study the motion of the double pendulum given various values for θ_1 and θ_2. An iterative method was used that cycled between all possible combinations of θ_1 and θ_2, given a specified interval of $\frac{\pi}{12}$. The program is set to run for 1000 seconds. Poincare sections are generated in each case for both the first and second pendulum. 156 plots are generated upon execution of the program. We can see from a sampling of plots that the motion of the double pendulum is *always* complex, and is often chaotic. A few of the plots do not fit the pattern of the rest but instead display erratic behavior- probably a signal of a limitation in the numerical method being utilized by the Mathematica kernel. Certain characteristic shapes can be seen to emerge with great frequency within the phase space of the Poincare sections, particularly the rounded diamond shape. There are varying degrees of complexity between the various plots, and without recourse to additional mathematical techniques, it is impossible to determine at exactly what point the hurdle is crossed from complicated motion to truly chaotic motion. However, a general idea of the level of complexity within the double pendulum system for various initial conditions can be seen quite easily in these plots. Below are a handful of examples pulled from the Mathematica program. Values for θ_1 and θ_2 are given, as well as whether the particular Poincare section represents the motion of the first or second pendulum.

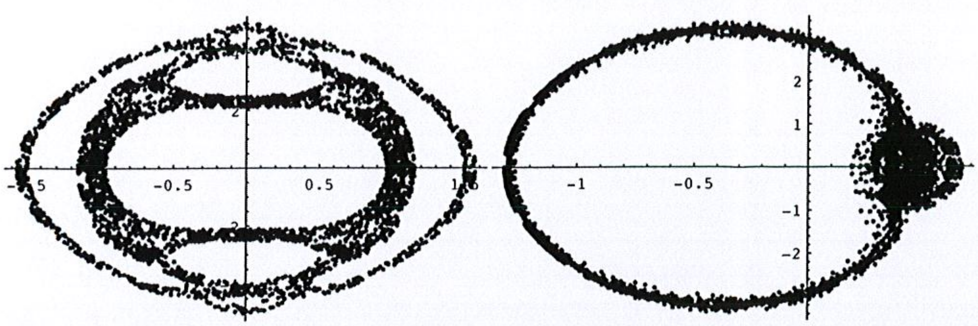

Figure 4: **Left - P_1 ($\theta_1 = -\frac{\pi}{2}$, $\theta_2 = -\frac{\pi}{6}$). Right - P_1 ($\theta_1 = -\frac{5\pi}{12}$, $\theta_2 = -\frac{\pi}{3}$).**

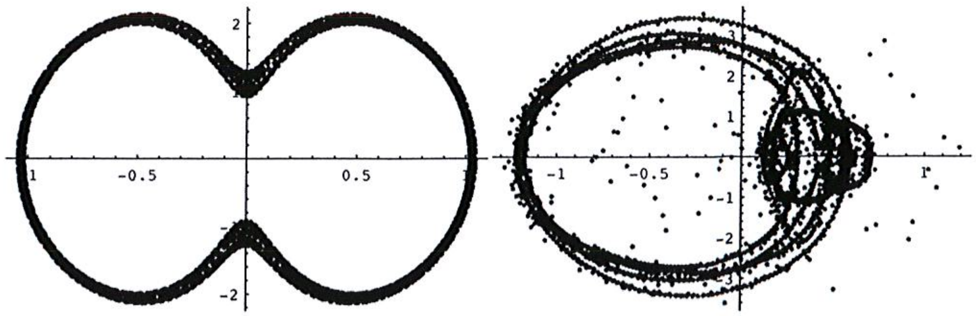

Figure 5: **Left -** P_1 ($\theta_1 = -\frac{\pi}{3}$, $\theta_2 = -\frac{5\pi}{12}$). **Right -** P_2 ($\theta_1 = -\frac{\pi}{4}$, $\theta_2 = -\frac{\pi}{2}$).

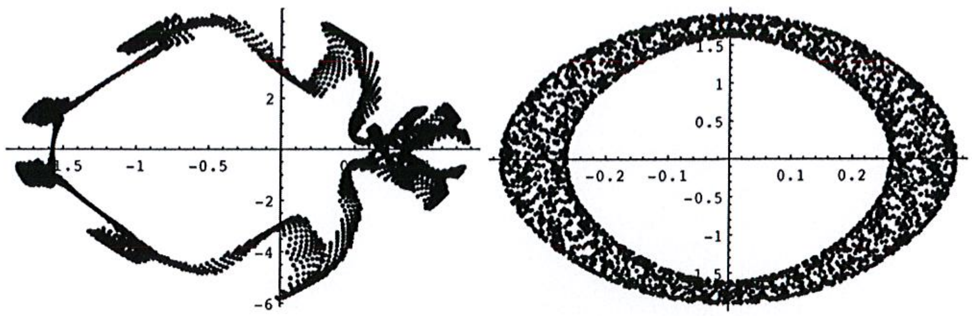

Figure 6: **Left -** P_2 ($\theta_1 = -\frac{\pi}{6}$, $\theta_2 = -\frac{\pi}{2}$). **Right -** P_2 ($\theta_1 = -\frac{\pi}{12}$, $\theta_2 = -\frac{\pi}{12}$).

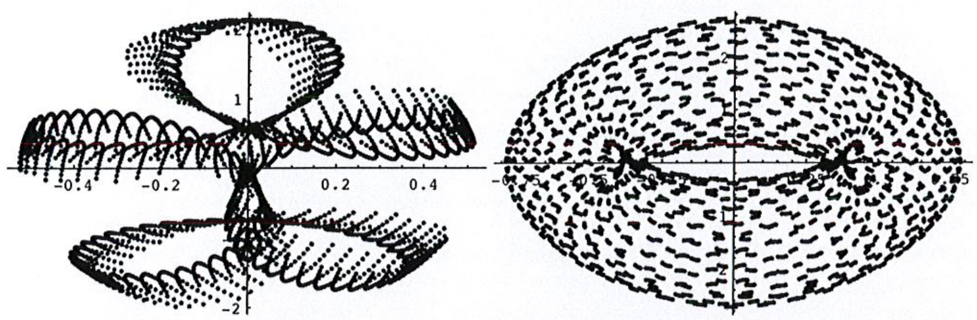

Figure 7: **Left -** P_2 ($\theta_1 = 0$, $\theta_2 = -\frac{\pi}{6}$). **Right -** P_2 ($\theta_1 = \frac{\pi}{12}$, $\theta_2 = \frac{\pi}{4}$).

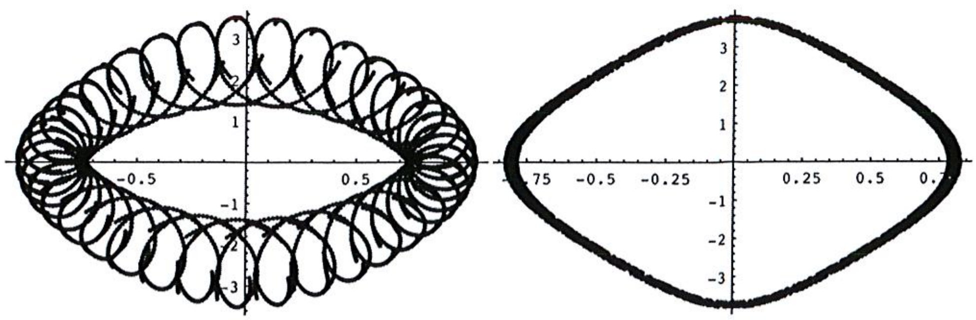

Figure 8: **Left -** $P_2\ (\theta_1 = \frac{\pi}{6}, \theta_2 = \frac{\pi}{3})$. **Right -** $P_2\ (\theta_1 = \frac{\pi}{4}, \theta_2 = -\frac{\pi}{4})$.

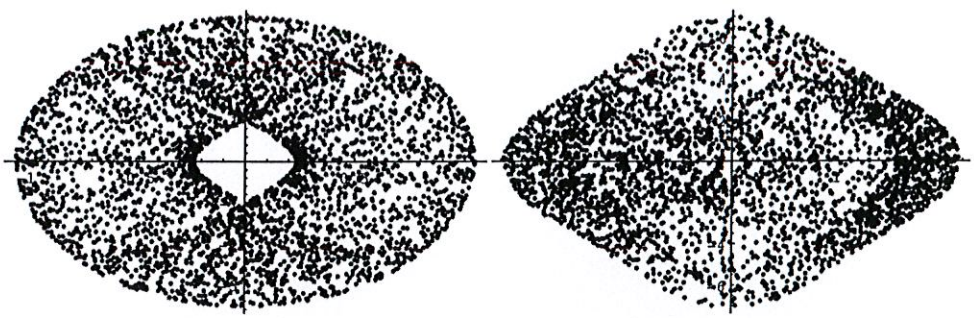

Figure 9: **Left -** $P_1\ (\theta_1 = \frac{\pi}{3}, \theta_2 = \frac{\pi}{12})$. **Right -** $P_2\ (\theta_1 = \frac{5\pi}{12}, \theta_2 = \frac{\pi}{4})$.

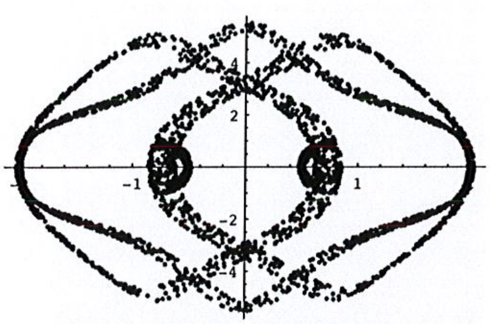

Figure 10: $P_2\ (\theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{6})$.

Future Work

Identification of chaos via Poincare sections is not an exact science. In general, we can know that when simple curves are present in the Poincare section, an analytic solution for the motion is possible, whereas when many complicated, irregular curves are present, we have chaos.[15] However, as we noted earlier, it is not always easy to see the difference between simple and irregular curves (in the case of the double pendulum, it *is* often easy to see). If we wish to quantify exactly where the points of chaos begin, i.e. where numerical solutions for the motion of the system are no longer practical, we must use Lyapunov exponents.[1] A detailed explanation of Lyapunov exponents is beyond the scope of this thesis, but would represent a logical next step in an analysis of the double pendulum.

Another method of determining visually where regions of chaos occur in a system is through of bifurcation diagrams.[15] Once again, a detailed explanation is not called for here. Suffice to say, a bifurcation diagram would essentially reduce the entire set of plots generated by our program to a single diagram showing where along the spectrum of values for θ_1 and θ_2 chaos occurs. The process of obtaining such a diagram is quite complicated however, and taken together with the determination of Lyapunov exponents, would present enough material for another thesis.

The formula by which Mathematica performs the numerical integration of the systems motions could also be studied in greater detail in an attempt to determine the limiting factors of the method. This could help explain the erratic behavior of certain solutions mentioned in the results section of this thesis.

Bibliography

[1] B., Davies,. Exploring chaos theory and experiment. Reading, Mass: Perseus Books, 1999.

[2] Bahr, David A. (1994) Self-similar and erratic transient behavior of the simple pendulum, Physics department student projects, vol. 4, 103-110.

[3] Baker, Gregory L. Chaotic dynamics an introduction. Cambridge [England]:

Cambridge UP, 1990.

[4] Corben, H. C. Classical mechanics. New York: Dover Publications, 1994.

[5] Gleick, James. Chaos making a new science. New York, N.Y., U.S.A: Viking, 1987.

[6] "Hands-on Start to Mathematica--Part 1." Wolfram Research: Mathematica, Technical and Scientific Software. 09 Apr. 2009 <http://www.wolfram.com/broadcast/screencasts/handsonstart/>.

[7] John, Briggs,. Fractals the patterns of chaos : a new aesthetic of art, science, and nature. New York: Simon & Schuster, 1992.

[8] Kilmister, C.W. Lagrangian Mechanics. London: Plenum Press, 1967.

[9] Kreyszig, Erwin. Advanced Engineering Mathematics, Textbook and Student Solutions Manual. New York: Wiley, 2006.

[10] Larson, Ron, Robert P. Hostetler, and Bruce H. Edwards. Calculus (With Analytic Geometry)(8th edition). Boston: Houghton Mifflin Company, 2005.

[11] "Mathematica Notebooks." University of Richmond. 09 Apr. 2009 <http://facultystaff.richmond.edu/~ggilfoyl/compphys/notebooks.html>.

[12] Schwalbe, Dan. VisualDSolve visualizing differential equations with Mathematica. Santa Clara, Calif: TELOS, 1997.

[13] Strogatz, Steven H. Nonlinear Dynamics and Chaos With Applications to Physics,

[14] "Surface of Section -- from Wolfram MathWorld." Wolfram MathWorld: The Web's Most Extensive Mathematics Resource. 09 Apr. 2009 <http://mathworld.wolfram.com/SurfaceofSection.html>.

[15] Thornton, Stephen T. Classical dynamics of particles and systems. Belmont, CA: Brooks/Cole, 2004.

[16] Virgin, Lawrence N. Introduction to Experimental Nonlinear Dynamics A Case Study in Mechanical Vibration. New York: Cambridge UP, 2000.

[17] Wolfram, Stephen. Mathematica book. Champaign, IL: Wolfram Media, Cambridge UP, 1999.

[18] Zill, Dennis G. A First Course in Differential Equations The Classic Fifth Edition.

Belmont: Brooks Cole, 2000.

Appendix

Mathematica code for generating Poincare sections used in this thesis:

*Solve the coupled ordinary differential equations for the given initial conditions.

```
Clear[poincarePend];
m1 = 1; m2 = 1; g = 9.81; L1 = 1; L2 = 1;
θ10 = -4 π/12; dθ10 = 0; θ20 = -π/2; tmax = 1000; steps = 1000000; wd = π/12;
poincarePend[m1_, m2_, g_, L1_, L2_, θ10_, dθ10_, θ20_, dθ20_, tmax_, steps_,
  wd_] :=
          temp1 = NDSolve[
                    {dθ1[t] - θ1'[t] == 0,
   dθ2[t] - θ2'[t] == 0,
   (m1 + m2)*L1*dθ1'[t] + m2*L2*dθ2'[t]*Cos[θ1[t] - θ2[t]] +
    m2*L2*dθ2[t]^2*Sin[θ1[t] - θ2[t]] + g*(m1 + m2)*Sin[θ1[t]] == 0,
   m2*L2*dθ2'[t] + m2*L1*dθ1'[t]*Cos[θ1[t] - θ2[t]] + g*m2*Sin[θ2[t]] == 0,
   dθ1[0] == dθ10, θ1[0] == θ10, dθ2[0] == dθ20, θ2[0] == θ20},
                    {θ1[t], dθ1[t], θ2[t], dθ2[t]},
                    {t, 0, tmax}, MaxSteps -> steps
                    ];
```

*Iterative structure: cycle through the various initial conditions given with the given interval and create Poincare sections for each combination of initial conditions.

```
For[i = 1, i < 2, i++;
          For[j = 1, j < 2, j++;
                    For[k = 1, k < 14, k++;
                              For [l = 1, l < 2, l++;
                              Print[
   SringForm["dθ1 v θ1 for (θ10, θ20 ,dθ10, dθ20 )", θ10, θ20, dθ10, dθ20]]
                              poincarePend[m1, m2, g, L1, L2, θ10, dθ10, θ20, dθ20, tmax, steps,
   wd];
                                        temp2 = Flatten[Table[{θ1[t], dθ1[t]} /. temp1, {t, 0, tmax, wd}], 1];
                                        ListPlot[temp2, PlotStyle -> PointSize[0.01], PlotRange -> All]
                              Print[
   SringForm["dθ2 v θ2 for (θ10, θ20 ,dθ10, dθ20 )", θ10, θ20, dθ10, dθ20]]
                              poincarePend[m1, m2, g, L1, L2, θ10, dθ10, θ20, dθ20, tmax, steps,
   wd];
                                        temp3 = Flatten[Table[{θ2[t], dθ2[t]} /. temp1, {t, 0, tmax, wd}], 1];
                                        ListPlot[temp3, PlotStyle -> PointSize[0.01], PlotRange -> All]
                              dθ20 = dθ20
                              ];
                    θ20 = θ20 + π/12
                    ];
          dθ10 = dθ10
          ];
θ10 = θ10 + π/12
];
```