LIMITS

Highlights From Over 2000 Years of
Developments in Calculus Limits

By:

Lauri A. Nevalainen

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Ask a calculus student to define a limit, and you may get answers such as, "Letting a number get really close to another" or "For every epsilon, there exists a delta,...and so on." Whatever their response, every calculus student could tell you that the process of taking a limit is fundamental to the subject of calculus. In fact, calculus, as we know it today, revolves around the process of taking a limit. Though mathematicians had been using calculus principles for more than a thousand years before its "discovery" in the 17th century, the formal epsilon-delta definition of a limit was not actually established until the 19th century.

This paper is intended to highlight the conceptual development of limits and the major mathematical contributions towards the establishment of the formal limit definition.

The Ancients

Zeno of Elea (ca 450 B.C.)

Little is known of the life of Zeno. Most of what is known is found in the writings of Plato. Zeno of Elea was known as a philosopher with a fondness for controversy. He wrote at least one major work, which gained him fame in the ancient world. His work is thought to have contained at least 40 paradoxes. Four of these paradoxes, taken from written works of Aristotle, have made significant impacts upon mathematics, but two in particular, t*he Dichotomy,* and *the Achilles,* have influenced thoughts about the infinite and the limiting process for thousands of years.

The Dichotomy argues that before a moving object can travel a certain distance, it first must travel 1/2 the distance. But before it can travel 1/2 the distance, it must travel 1/4 of the distance, and before it can travel 1/4 of the distance, it must travel 1/8 of the distance, and so on, resulting in an infinite number of subdivisions. Considering this situation, one realizes that motion could never occur, as it would be impossible to begin movement.

Another of Zeno's paradoxes, *the Achilles,* also deals with movement and an infinite number of subdivisions. According to Zeno, should the fast runner Achilles give the slower tortoise a head start in a race, Achilles would never be able to catch the tortoise, because in the time it takes Achilles to reach the original starting point of the tortoise, the tortoise would have progressed some distance further. Then, by the time Achilles covered this distance, the tortoise would have progressed even further, and thus Achilles would never catch the tortoise, no matter how fast he is or how slow the tortoise is. (Smith, 1996)

These paradoxes were very perplexing to many in Zeno's time, as they were for thousands of years after. It is because of these that many mathematicians and scientists shied away from the infinite. Zeno brought about many questions and mysteries concerning the infinite, and it was not until the time of Newton, more than 2,000 years later, that mathematicians would start to work comfortably with infinite number series.

Archimedes (ca 287-212 B.C.)

Archimedes was born in the Greek city of Syracuse, on the island of Sicily. In his lifetime, he is credited with writing 9 treatises consisting of his own discoveries. Included in these treatises is his "Method" for finding surface areas and volumes. Archimedes intuitively divided geometric figures into smaller figures of lesser degree. Basically, he considered surfaces to be "made-up" of an infinite number of parallel lines. He considered revolution solids to be "filled up" by circles. Archimedes never proved such methods, and thus he knew his "Method" lacked rigor. He did not consider intuitive reasoning to be proof, only a stepping-stone for a more rigorous form of exhaustion, though his "Method" included many concepts common to present day calculus textbooks. (Boyer, 1989)

Archimedes wrote *Quadrature of the Parabola (*exact date unknown). (*Quadrature* is the act of finding an area.) In this work, Archimedes calculated the area of the part of the parabola bounded by an arbitrary chord QQ'. (Figure 1)

In calculating the area, Archimedes systematically inscribed an infinite number of triangles, starting with QPQ', where P is the point at which the tangent to the curve will be parallel to QQ'. Then R and R' were chosen so that the tangents to the curve were parallel to QP and Q'P respectively. The process of choosing points and drawing triangles could continue infinitely.

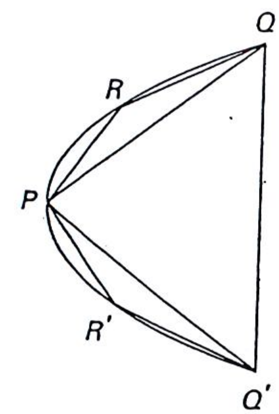

Figure 1
Archimedes' method of quadrature.

With commonly known geometric methods, Archimedes showed that the sum of the areas of QRP and Q'R'P equals 1/4 the area of QPQ'. He then calculated that the next 4 triangles would sum to 1/4 of the sum of QRP and Q'R'P or 1/16 of QPQ'. The next triangles would sum to 1/64 QPQ', and so on. Summing the area of the infinite number of triangles (1+1/4+1/16+1/64 +...) QPQ' is a simple task for mathematicians today, as we know the formula for the sum of an infinite geometric series, $1 + 1/r + 1/r^2 + \ldots$ is $1/(1-r)$ where $-1<r<1$. In other words, in this case, $1+1/4+1/16+1/64+\ldots=1/(1-1/4)=4/3$. Clearly, in his time though, Archimedes did not have the formula to sum an infinite geometric series, but he did look at this sum geometrically, and Archimedes reasoned that the more triangles summed, the closer the area became to 4/3 QPQ'. Thus, Archimedes proved the area of a parabola bounded by an arbitrary chord is equal to 4/3 the area of the triangle QPQ'. (Calculating the area of triangles was a familiar process at that time.) (Simmons, 1992)

Archimedes did not have a formal definition of limit, but he showed that he was subtly aware of the concept. It is because of thinking such as this that Archimedes is considered one of the greatest geniuses of the Ancient World. (Burton, 1985)

The ideas of infinity and limitless numbers were ideas most mathematicians avoided, as they had no way to get a firm grasp on the concepts. This was one of the reasons no further significant progress towards the concept of limits was made for more than a thousand years. In addition to the uncertainty of the infinite, European mathematics entered a very bleak period from 476-1000 A.D., known as the Dark Ages. During this period, virtually no mathematical or scientific advances were made, as living conditions were poor. Though the standard of living improved and the Dark Ages phased out around the year 1000, it was not until the 17th century that a hint of limits crept back into mathematicians' thoughts.

17th & 18th Century

This period is best characterized by the use of infinitesimals--small quantities considered to be smaller than any non-zero real number. Mathematicians of this period became more comfortable working with the infinite, thus allowing connections to be made. Many advances were made in the mathematical world, including the discovery of analytical geometry and calculus, but mathematicians still struggled with the concept of a limit. Many criticized and were criticized for vagueness in dealing with the infinite. The subject of calculus was developed during this period, yet there still is no trace of a limit definition, as we know it today. Mathematicians worked intuitively, not looking for exactness; results were valued above exact methods.

Bonaventura Francesco Cavalieri (1598-1647)

Bonaventura Cavalieri grew up in present-day Italy, and it was there he did his mathematical work. He considered himself a disciple of Galileo, and the two corresponded regularly to discuss mathematics and science. Cavalieri also corresponded with other mathematicians such as Mersenne , Toricelli, and Kepler. He was considered one of the greats of his time.

In 1635, Cavalieri published his most famous work, *Geometria indivisibilis continuorum nova.* This work was a combination of the methods of Archimedes and theories of Kepler for finding areas and volumes. In his work, Cavalieri described his "Method of Indivisibles," which basically involved dividing geometric figures into smaller pieces, though he did not use triangles as earlier mathematicians had. Cavalieri considered geometric figures to be made up of other figures of lesser degree. He imagined every area to be composed of an indefinite number of parallel lines and every volume to be composed of an indefinite number of plane sections. He called these lines and planes indivisibles. Summing these indivisibles (summing an infinite number of terms) allowed for an easy, simple way to calculate areas and volumes. (Malet, 1996)

The concept of dividing an area into many smaller figures was not entirely new to 17th century mathematicians, as Cavalieri's method was very similar to that of Archimedes. His method was very geometric, and many mathematicians were opposed to his ideas. There definitely were problems with Cavalieri's "Method." First of all, the notion of summing an infinite number of terms and the development of rules involving this sum were difficult concepts for mathematicians to grasp. But for the most part, this difficulty did not bother mathematicians nearly as much as the lack of a clear notation and a definition for indivisibles. Terms like "all of the lines" and "all of the planes" were used by Cavalieri to describe his "Method," but these were not concrete enough definitions for most. (Simmons, 1992)

In Cavalieri's defense, indivisibles made good sense, and few mathematicians argued against the concepts behind his "Method." The lack of a rigorous definition and notation caused the consternation. Indivisibles were a powerful tool, and Cavalieri used them to prove theorems by Euclid, Archimedes, and other mathematicians. The lack of rigor did not bother Cavalieri, for he was convinced of his method and the results it showed. Due to the attacks by others, in 1647 Cavalieri published *Exercitationes geometricae sex,* a "second edition" of his first work. This work became a main source book for mathematicians of the 17th century. (Burton, 1985)

Much of Cavalieri's work involved analytic geometry and calculus, neither of which had been fully developed at the time. He had no formal knowledge of limits, though his "Method" showed many signs of a working understanding. When possible, Cavalieri actually avoided the ideas of the infinitely large and small, thus the lack of rigor in his work. Despite his avoidance of the infinite, Bonaventura Cavalieri made a mark on the development of limits, a topic quite synonymous with the infinite.

In addition to Cavalieiri's "Method of Indivisibles" other 17th century mathematicians, including Kepler, Fermat, Leibniz, and a reluctant Newton, used a similar method of "infinitesimals". An infinitesimal is considered a value smaller than any real number value. In calculating areas, the "Method of Infinitesimals" involved dividing an object into other geometric objects of the same degree. For example, planes were divided up into a large or infinite number of very small (infinitesimal) planes. Infinitesimals eventually replaced indivisibles. In fact, Leibniz based his calculus on infinitesimal quantities.

Pierre de Fermat (1601-1665)

Pierre de Fermat, a 17th century French mathematician, made significant contributions towards the concept of calculus limits, though calculus and limits had not yet been defined. Though he used many methods and made discoveries, he was very unmotivated to publish his findings. Fermat enjoyed pure mathematics; he did not take such pleasure in practical applications. In working with these applications, Fermat was sluggish and unclear in his writing, though his concepts were often correct. It is for this reason that his techniques of finding tangents to a curve as well as maximum and minimum points were not readily accepted.

Fermat is credited with many advances in analytic geometry. Using his analytic geometry, he found the equations of familiar curves, and he constructed many new curves.

In working with these curves, Fermat investigated maximum and minimum points and in doing so, applied a neighborhood process. In essence, Fermat considered a point (a, f(a)) on f(x) and a neighboring point (a+E, f(a+E)). Normally, these points would be quite different, but at the top or bottom of a smooth curve, Fermat discovered they were much closer than expected. He then reasoned that choosing smaller values of E would bring f(a) and f(a+E) so close that in fact, they could virtually be considered equivalent. Fermat then subtracted the two, divided by E, and then set E=0. This resulted in the abscissas of the maximum or minimum point of a polynomial curve. Today we know this method as the process of differentiation, except E has generally been replaced with h or Δx. The act of differentiation is strongly linked with limits, yet Fermat had no working definition of a limit in the 1600s. (Boyer, 1989)

Later in his work, Fermat became curious about the "problem of tangents." He realized that he could apply his techniques for finding maximum and minimum points towards finding tangents to a curve. Fermat considered a point (a,f(a)) and a neighboring point (a+E, f(a+E)). If E was allowed to get smaller and smaller, then (a+E, f(a+E)) would lie closer and closer to (a,f(a)) and could be viewed as lying on both the curve and the tangent. He then used this result to show that the slope of a tangent line could easily be calculated. Fermat commented that this process was very similar to his process for finding minimum and maximum, and therefore he did not need to provide the details. It is because of this many respected mathematicians, such as Descartes, did not find his method valid.

Fermat also developed a procedure for finding the area under curves. For example, in finding the area under $y = px^{-k}$, k>1, Fermat partitioned the x-axis into infinitely many segments with lengths equivalent to the terms of a geometric progression, which he could easily sum. Using (m/n) as his geometric ratio, he divided the axis to the right of x_0 at points $a_1=(m/n)x_0$, $a_2=(m/n)^2x_0,\ldots$, where m and n are positive integers with m>n. (Figure 2.)

Next, Fermat constructed rectangles from the lengths, and the first rectangle R_1 has area

$$R_1 = (\frac{m}{n}x_0 - x_0)y_0 = (\frac{m}{n} - 1)x_0\frac{p}{(x_0^{\,k})} = (\frac{m}{n} - 1)\frac{p}{(x_0^{\,k-1})}.$$

The next rectangle R_2 has area:

$$R_2 = [(\frac{m}{n})^2 x_0 - (\frac{m}{n})x_0]\frac{p}{(\frac{m}{n}x_o)^k} = (\frac{m}{n})(\frac{m}{n}-1)x_0(\frac{n}{m})^k\frac{p}{(x_o^k)} = (\frac{n}{m})^{k-1}R_1.$$

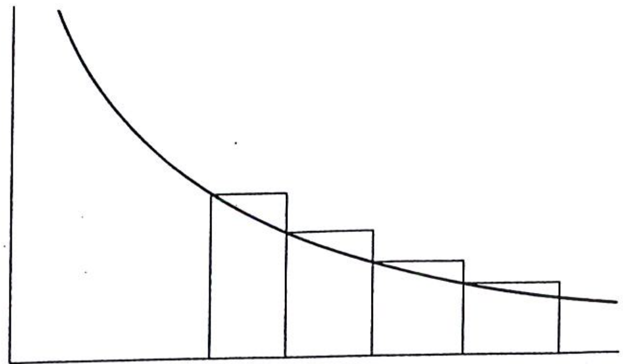

Figure 2
Fermat's method for finding the area under the curve $y=px^{-k}$

Likewise, the third rectangle has area $R_3 = (\frac{n}{m})^{2(k-1)}R_1$, and summing these rectangles is equal to

$$R = R_1 + (\frac{n}{m})^{k-1}R_1 + (\frac{n}{m})^{2(k-1)}R_1 + ... = R_1[1+(\frac{n}{m})^{k-1}+(\frac{n}{m})^{2(k-1)}+...].$$

Using the formula for the sum of a geometric series,

$$R = \frac{1}{1-(\frac{n}{m})^{k-1}}R_1 = \frac{1}{1-(\frac{n}{m})^{k-1}}(\frac{m}{n}-1)\frac{p}{(x_o)^{k-1}} = \frac{1}{\frac{n}{m}+(\frac{n}{m})^2+...+(\frac{n}{m})^{k-1}}\frac{p}{(x_o)^{k-1}}.$$

Next, Fermat let the area of the first rectangle get infinitely small or "go to nothing" by letting (n/m) get closer and closer to 1. R then gets closer to $(\frac{1}{k-1}\frac{p}{(x_o)^{k-1}})$, and so the area under the curve $y=px^{-k}$ is given by $A = \frac{1}{k-1}x_0y_o$, $k>1$.

Being familiar with calculus, one may notice that Fermat basically knew concepts from the subject, which was not yet discovered. Though Fermat did understand how to partition an interval to find an area, he did not make the connection between that and tangents to a curve, which leads to the fundamental theorem of calculus, something both Newton and Leibniz realized.

Fermat used a "limiting process" on a regular basis. His neighborhood process would later prove very applicable when considering a formal definition of a limit, but not for hundreds of years.

Isaac Newton (1642-1727)

Isaac Newton, an Englishman, is known as one of the founders of calculus, though he is quoted as saying, "If I have seen farther than [others], it is because I have stood on the shoulders of giants." (O'Connor, 1996) Newton was very familiar with the works of other mathematicians such as Kepler, Galileo, Fermat, and Descartes.

In the years 1665-1666, Newon, not yet 25, developed his calculus, though his results were not published until 1687. He was a very secretive, inward man who had no desire to publish his findings. Newton used developments of his teacher, Isaac Barrow upon which to build his own results. Barrow researched drawing tangents to curves and determining areas bounded by curves, and he printed his results in *Lectiones Geometricae*, a series of lectures he presented at Cambridge University. Barrow considered problems of distance and velocity and discussed the inverse relationships between the two. Newton used this line of thinking to develop calculus. (Burton, 1985) He analyzed flowing quantities, which he referred to as "fluents" and rates of change, which he referred to as "fluxions." Newton developed calculus as a very practical form of mathematical computation. (Boyer, 1989)

Newton basically realized the concept of a limit, but in calculating ratios, he let very small quantities "vanish." He just disregarded very small terms. His informal concept of the limiting process is found in one of his publications, *Philosophiae naturalis principia mathematica*, the most admired scientific treatise of all times. Lemma I of this work states:

> *Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer to each other than by any given difference, become ultimately equal.* (Boyer, 1989)

Newton linked his "fluents and fluxions" with problems of infinite series and in doing so developed what he called, "My Method," a method now known as the "Method of Fluxions." He used infinite series in ways similar to the ways finite polynomials were used. He proposed that infinite series had the same consistency as finite quantities and that they shared the same laws. He did not use infinite series as an approximating tool as other mathematicians before him had. He used them as alternate forms of functions, and he encouraged others to use them this way as well. Newton also developed a series expansion for the sine and cosine functions. Until this point, mathematicians had been wary of infinite series, but Newton worked to change these thoughts. (Boyer, 1989)

Gottfried Wilhelm von Leibniz (1646-1716)

Gottfried Wilhelm von Leibniz, a German thought to have an IQ of 180 or higher, was considered more of a philosopher than a mathematician. He, along with Newton, is credited with the development of calculus. Though his methods came later than Newton's, Leibniz was much more willing to publish his works, and thus his calculus was published first. The two maintained a friendly correspondence early in their careers, but the publishing of Leibniz's calculus caused a bitter quarrel, and the two came to despise each other. (Simmons, 1992)

Though Newton and Leibniz fought, their ideas were very similar. Each built his calculus upon the ratios and products of infinitely small quantities (infinitesimals); Newton called such quantities fluxions, and Leibniz called them differentials. Both used theories of infinitesimals in developing their analysis.

It is said that while studying one day, a light came upon Leibniz, and he realized that a tangent to a given curve can be found by taking the ratio of the differences in the abscissas of two neighboring points on a curve as these differences become infinitely small. (Simmons, 1992) He knew Fermat's earlier methods of finding area by summing up the infinitely thin rectangles (infinitesimals) making up this area. He observed that by drawing and summing these infinitely small rectangles under a curve, one also must consider the small "infinitesimal triangles" that are formed between the top of the rectangle and the curve.

Leibniz's infinitesimals were "there, but not there." Though there were flaws in his method, there were also many useful concepts. The beauty of Leibniz's method was that it could be used with any function; it was very general in form. Leibniz is also well known for his precise notation. The simple notation he used in developing his mathematics in the 17^{th} century is basically the same notation used today.

Throughout the 18th century, mathematicians willingly used the power of the newly developed calculus to bring about many results. Calculus worked, and few concerned themselves with why; few questioned infinitesimals, the base of 18th century calculus. Many considered the results most important, and calculus led to powerful results. There was, however, overwhelming doubt as to whether calculus was truly legitimate. Many mathematicians did not feel there were significant developments and proofs to accept the concepts of calculus as valid. Despite criticism, many mathematicians continued to work towards a more clear, rigorous, calculus.

Brook Taylor (1685-1731) developed a method for the expansion of a function about a point. He reasoned that any continuous function could be written as the sum of polynomial functions. In general if f(x) is a continuous function, with at least n+1 derivatives, then:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n)}(a)(x-a)^n}{n!} + R_n$$

where $f^{(n)}$ (a) is the nth derivative of f(x) evaluated at a center "a," and R_n is a remainder function. (Taylor did not originally include this remainder function. It was later added by Joseph-Louis Lagrange, who calculated that a remainder function was necessary to make the expansion accurate.)

In 1797 Joseph-Louis Lagrange (1736-1813), published *Théorie des fonctions analytique,* in which he produced his theory of real functions. The goal of this work was to provide:

> *... the principles of the differential calculus, freed from all consideration of the infinitely small or vanishing quantities, of limits or fluxions, and reduced to the algebraic analysis of finite quantities.*

Basically, Lagrange believed he could do this using Taylor's series expansions. Lagrange theorized, as many before him did, that every function could be written as a power series, specifically a Taylor expansion, and then integration and differentiation would be possible term-by-term. Mathematicians of this period believed that the integration of a series was equal to the sum of the term integrals. It was also thought to be the case that if an infinite series converges on some interval, to a continuous, differentiable function, then the differentiation of that series term-by-term would also converge. Several mathematicians, including Joseph Fourier, showed this was not always the case.

Frenchman Jean Baptiste Joseph Fourier (1768-1830) published his famous work *Theorie analytique de la chaleur* in 1822, when he developed a mathematical theory of heat. Fourier showed that any piecewise defined function, either continuous or discontinuous, could be represented as a Fourier series, an expansion of sines and cosines of multiples of the variable. Fourier used the following as his expansion function:

$$f(x) = \frac{a_0}{2} + \sum_{j=1}^{\infty} [a_j \cos(jk) + b_j \sin(jx)].$$

In fact, Fourier discovered a piecewise continuous function which was differentiable, but term-by-term differentiation of its expansion series did not converge to the derived function. For example, the expansion

$$\frac{1}{2}x = \sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots.$$

$$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

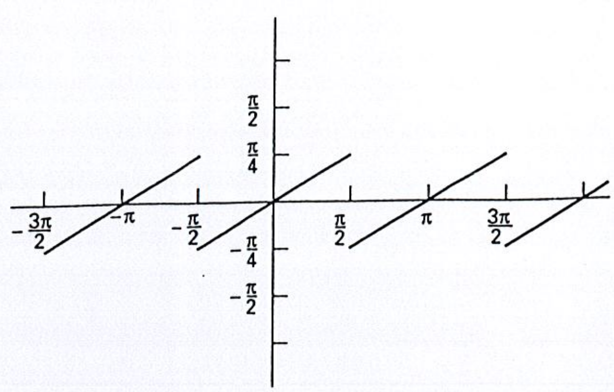

Figure 3
Graph of the Fourier expansion
y= sin x - 1/2sin2x + 1/3sin3x – 1/4sin4x+…

Up until this point, mathematicians had used differentiation, integration and the rearrangement of series without questioning its validity. Calculus had brought about so many results to practical problems in the past, but the previous confidence was shattered as many realized that something was missing. Not every series converged as Taylor series did. Mathematicians realized that differentiation and summation were not always interchangeable. It was because of this that 19th century mathematicians were forced to take a closer look at the act of differentiation and the concept of limits.

19th Century

This period is one of refinement and rigor, as calculus took on a more precise, exact form. A limit process and the concept of being "close" to a number replaced the former ideas of infinitesimals. Many mathematicians contributed to the addition of rigor to calculus, but two particular mathematicians led all others in the development and critiquing of ideas.

Augustin-Louis Cauchy (1789-1857)

Augustin-Louis Cauchy, a native of France, was the most published mathematician in the 19th Century; he wrote 8 full-length books and 789 papers, equivalent to 26 large volumes. He established a private journal to publish his vast number of works. Cauchy has his name attached to 16 concepts and theorems, more than any other mathematician.

One of Cauchy's most famous works is *Cours d'Analyse de l'Ecole Royale Polytechniques* of 1821. This work had a major impact on the understanding of continuity, limits, integrals and convergence. Unlike Lagrange, Cauchy realized that calculus could not be handled without the use of some limiting process. Earlier mathematicians considered infinitesimals to be small fixed numbers but Cauchy redefined infinitesimals as limits of dependent variables.

> *"One says that a variable quantity becomes infinitely small when its numerical value decreases indefinitely in such a way as to converge toward the limit zero."* ( )

Cauchy's *Cours d'Analyse* contained the definition of limit that would be used until 1870 when the modern epsilon-delta definition was developed. His definition is as follows:

> *When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last is called the limit of all others.* ( )

Cauchy thus reformatted calculus in terms of point-wise limits rather than infinitesimals, which had been the base of calculus since the time of Newton and Leibniz. Cauchy replaced the need for small quantities with the concept of "being close." He redefined integrals as the limit of a sum, rather than the antiderivative, and he used limits in differentiation as well.

In addition to a formal definition of limit, Cauchy addressed the issues of continuity and convergence. He believed that:

> *When the different terms of the series are functions of the same variable x, continuous with respect to that variable in the neighborhood of a particular value for which the series is convergent, the sum s of the series is also, in the neighborhood of this particular value, a continuous function of x.* ( )

Cauchy dealt with limits and convergence of an infinite series in a point-wise manner. Basically, he reasoned that for a sequence of functions $f_n(x)$ defined on a domain D, the sequence will converge to a function F also defined on D, if for any x in D, the sequence $f_n(x)$ converges to F(x), where N_o is dependent on x as well as ε:

$$\forall x \in D, \forall \varepsilon > 0, \exists N_0 \left| n \geq N_0 \Rightarrow \left| f_n(x) - F(x) \right| < \varepsilon. \right.$$

Cauchy did not know what we now know about uniform convergence. Cauchy believed that the limit function of a convergent sequence was continuous. (This was proved incorrect in 1826 when Fourier's expansion sin x – ½ sin(2x) + 1/3 sin(3x)-... was shown to be not continuous at x=(2n-1)(π/2), n is an integer)

In *Cours d'Analyse,* Cauchy developed a concrete definition of convergence based on limits:

> Let $s_n=u_0+u_1+u_2+\ldots+u_{n-1}$ be the sum of the first *n* terms [of a series], *n* designating an arbitrary integer. If, for increasing values of *n*, the sum s_n approaches indefinitely a certain limit s, the series will be called convergent, and the limit in question will be called the sum of the series. If as *n* increases indefinitely the sum s_n does not approach any fixed limit, the series will be divergent and will not have a sum." ( )

Cauchy realized that in order for a series to converge, its terms must decrease towards zero, but this alone was not a guarantee of convergence. Cauchy also stated that in order for a series to converge, its partial sums must always be less than some assignable value. He never proved this, but he did give examples. Cauchy wrongly believed that if an infinite series converges for some interval, to a continuous, differential function then differentiating that series term by term would also converge. (He failed to consider convergence of the derived function.)

Cauchy sought to add clarity and rigor to calculus, and he is credited with being the first mathematician to do just that. He eliminated infinitesimals, and formed calculus into a system of precise theorems of convergence, continuity, derivatives, integrals and limits.

Karl Weierstrass (1815-1897)

Though Cauchy began the rigor in calculus, German teacher and mathematician Karl Weierstrass continued to add even more exactness to the subject. He was extremely careful in his reasoning, and he worked to eliminate the remaining vagueness in the basic concepts of calculus. Weierstrass is responsible for the formal epsilon-delta definition of a limit. Cauchy's use of terms such as "approaches indefinitely", "infinitely small increase", and "as little as one pleases" still lacked exactness, and Weierstrass removed any use of vague terms by dealing with limits and a neighborhood process. Weierstrass rigorized Cauchy's definition of limit, writing:

$$\lim_{x \to c} f(x)=L \text{ if for any given } \varepsilon > 0 \text{ there is a } \delta > 0$$

$$\text{such that } |f(x)-L| < \varepsilon \text{ whenever } |x-c| < \delta.$$

Weierstrass then went on to redefine limits uniformly, rather than point-wise. He looked at sequences of functions that were continuous but the limiting function was not. For example:

$f_n(x)=x^n$, $0 \leq x \leq 1$, $n \geq 1$.

This function is point-wise convergent as Cauchy defined, it is continuous for all values of n, but the limiting function F(x)= 0 when $0 \leq x < 1$, but F(x)=1 when x=1. Thus, F(x) is not continuous, and Weierstrass had to take a closer look at convergence.

Cauchy defined convergence with N_0 depending on both ε and x, but Weierstrass now defined N_0 to only be dependent on ε:

$$\forall \varepsilon > 0, \exists N_0, \forall x \in D \left| n \geq N_0 \Rightarrow \left| f_n(x) - F(x) \right| < \varepsilon. \right.$$

He then used this to clarify convergence. He showed that in order for integration and differentiation to be commutative with summation, uniform convergence must exist: to guarantee differentiation and summation can be interchanged, one must first be sure the derived series will converge. Weierstrass reasoned that only when uniform convergence is present would the interchanging of limits with integration or differentiation with a series be possible. This is why Taylor expansions were differentiable term-by-term, and Fourier expansions were not. Taylor series have uniform convergence, and Fourier series do not.

Weierstrass worked, as Cauchy did, to develop a rigorous, precise calculus. His work with continuity and limits added the clarity needed for mathematicians to take analysis to the next level. Weierstrass in considered the world's greatest analysist during the latter part of the 19th century. He was the "father of modern analysis."

From Zeno's first thoughts of the infinite, to Weierstrass' epsilon-delta definition, calculus limits have evolved immensely in the past 2500 years. Without the development of the limit, there would not have been such great developments in analysis. Limits are a very important piece of calculus, and I believe understanding their development increases one's understanding of calculus.

I really enjoyed researching the topic of limits. I found the early years in the history of limits to be fascinating. I was intrigued with the fact that the epsilon-delta definition of a limit, as I know it, was developed less than 100 years ago, though calculus was developed over 300 years ago.

I found the later portions of limit history to be rather frustrating. I really enjoyed the learning I did, but I found some of the topics were a bit more complex than I had first perceived them to be. There is definitely more involved with the topic of limits than I first knew. I realize that I need to continue my education by taking an Analysis course in the future. I am excited to learn more about the topics I have discussed here, and I hope the opportunity to take an Analysis class presents itself soon.

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