Orthogonal Curvilinear Coordinates

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ABSTRACT

Mathematics is frequently discussed in the Cartesian coordinate system, but this system has its downfalls. In most cases, it does not conveniently represent the curved shape of the physical world. Gauss saw the need for a general coordinate system that would allow one to explore the properties of curved surfaces. It was only then that the curvilinear coordinate system was born. With Riemann's insight, it grew to represent infinite space, and curvilinear coordinate systems were utilized in physical applications. Orthogonal curvilinear coordinates, in particular, were used in solving select partial differential equations, including the Laplace and Helmholtz equations. The focus of this study was restricted to the derivation and application of orthogonal three-dimensional coordinate systems.

A curvilinear coordinate system expresses rectangular coordinates x, y, z in terms of the generalized coordinates u_1, u_2, u_3. By holding u_2 and u_3 constant, we form a family of u_1 surfaces. Similarly, we can form families of u_2 and u_3 surfaces. A coordinate system is orthogonal if the three families of coordinate surfaces are mutually perpendicular.

There are more than fifteen three-dimensional orthogonal curvilinear coordinate systems of degree two or less. Throughout the following research and application, four of these systems are considered: cylindrical coordinates, spherical coordinates, elliptic cylindrical coordinates, and parabolic coordinates.

1. HISTORY OF COORDINATE SYSTEMS

The idea of a coordinate system has existed for centuries. In Egyptian hieroglyphics, the symbol for land that has been surveyed is a cross hatching of vertical and horizontal lines. Towns in ancient Rome were laid out on a rectangular plan with two principle streets running to cardinal points on the compass. The Greeks, who are credited with "inventing" mapmaking, were the first to use coordinates in geography. In 320 BC, Dicæarchus of Mesina, a disciple of Aristotle, created a map of the world referencing two axes directed to the cardinal point of a compass.

As the idea of a coordinate system evolved, many respected mathematicians contributed to the development of curvilinear coordinates. Nicole Orseme (1323-1382) studied the distance covered by an object moving with variable velocity. He associated the instants of time within the interval with points on a horizontal line segment, similar to a modern x-axis. At each of these points, he erected a vertical line segment, the length of which represented the speed of the object at the corresponding time [3]. We would now call Orseme's depiction the graph of a function. Orseme then understood that the area under his graph represented the distance covered, for it was the sum of all the increments of distance corresponding to the instantaneous velocities.

Frenchmen Pierre de Fermat and René Descartes independently explored the idea of analytic geometry, also known as coordinate geometry. They were the first to truly unite algebra and geometry. Fermat's findings, although rarely published, dealt with the geometry of analysis and of infinitesimals. He realized that from the "specific property", or equation of a curve, all of its properties could be deduced. From ideas he began contemplating in 1629, his later papers explored the application of infinitesimals to the determination of the tangents to curves and the questions of maxima and minima.

Unlike Fermat, who had few public works in his lifetime, Descartes published many of his works in the theory of analytic geometry. Although "Cartesian coordinates" and the application of algebra to geometry had already appeared before he published La Géométrie in 1636, the book's merit lies in its application of 16th century algebra to ancient geometry. Descartes had an idea that a curved line could be expressed algebraically in terms of an equation involving the perpendicular distance of a general point to two perpendicular

reference lines. He is credited with being the first to make a graph, allowing a geometric interpretation of a mathematical function, and giving his name to Cartesian coordinates.

During the 1700's, mathematics progressively led to Gauss' discovery of curvilinear coordinates. Gottfried Leibniz introduced mathematical terms like "coordinates", "abscissa" and "ordinate". In 1731, Antoine Parent found the modern equation of a sphere, and Alexis Clairaut studied curves of double curvature. Clairaut published a treatise on these curves in three-dimensional space and recognized these curves as the intersection of surfaces.

When Carl Friedrich Gauss introduced curvilinear coordinates, he made a significant advancement in coordinate geometry. Gauss did not discover this general coordinate system by accident; rather, his theory was driven by inner necessity. In the early 1800's, the Hanoverian government requested Gauss to participate and lead a geodetic surveying of their land. There were patches of hilly countryside where Cartesian coordinates could not be used. Gauss faced the practical problem of making measurements on a curved surface.

Curved surfaces were considered before Gauss. The equation of a curved surface was given in the form $z = f(x, y)$ or $F(x, y, z) = 0$. Gauss conceived the idea that it was "not fair" to consider natural surfaces as well circumscribed portions of the Cartesian three-dimensional space. "The surface has its own properties and one should be able to investigate these properties without leaving the surface [7]." Gauss' idea was to draw two sets of arbitrarily chosen mutually intersecting lines as coordinate lines of the surface, given that the lines comply with the usual continuity and differentiability conditions. The coordinate lines introduced a pair of numbers u, v-called curvilinear or Gaussian coordinates-by which the points of the surface could be characterized [7].

A point P on the surface now became an intersection of the coordinate lines u and v. The rectangular coordinates x, y, z could be expressed as functions of u, v:

$$x = x(u, v) \qquad y = y(u, v) \qquad z = z(u, v)$$

This "parametric" representation of a surface became of greatest importance in geometry and physics. Gaussian coordinates' flexibility sparked new geometrical research and became a popular mathematical

tool in astronomy, geology, and eventually quantum mechanics.

The introduction of curvilinear coordinates was only one small element in Gauss' geometrical theory. Gauss extended the concept to create a truly new approach to geometry, leaving Euclid's geometry far behind.

The basic construction elements of Euclid's geometry are points, straight lines, angles, and circles. Gauss demonstrated that he could "erect the entire edifice of geometry" from only one postulate: the distance between two points (x_1, y_1) and (x_2, y_2) :

$$s^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

Ultimately this one postulate leads to the basic distance expression:

$$ds^2 = dx^2 + dy^2$$

We call ds the "line element". Gauss found that the same results were obtainable in Gaussian coordinates u, v. He went much further than Euclid in his investigation, for Gauss showed that the same construction elements which operated on a curved line were equally available for a curved surface. Thus the line element

$$ds^2 = dx^2 + dy^2 + dz^2$$

We call the metrical geometry, generated on a curved surface by the line element, the intrinsic geometry of the surface. "Straight lines" now mean "shortest lines" *measured on the surface*, and are often called "geodesics" to avoid confusion.

Gauss' discovery of curvilinear coordinates was extended to infinite dimension by Bernhard Riemann throughout the 1830's and 1840's . Riemann was often shadowed by his successor, and he did not feel that his work merited either attention nor publication, yet it was Reimann's discovery of tensors that later became the mathematical crux of Einstein's Theory of General Relativity. Einstein wanted to formulate the laws of physics in arbitrary coordinates, provided that space and time are united into one single four–dimensional geometry. From the principles of tensor calculus, Einstein realized that this was physical in nature, but geometrical at heart.

2. DERIVING ORTHOGONAL CURVILINEAR COORDINATES

The properties of curvilinear coordinates can be generically derived and then applied to each orthogonal coordinate system. In this section, we will derive the line element, the element of volume, the gradient, the divergence, the curl, and the Laplacian given the transformation from three–dimensional Cartesian coordinates to another three–dimensional system.
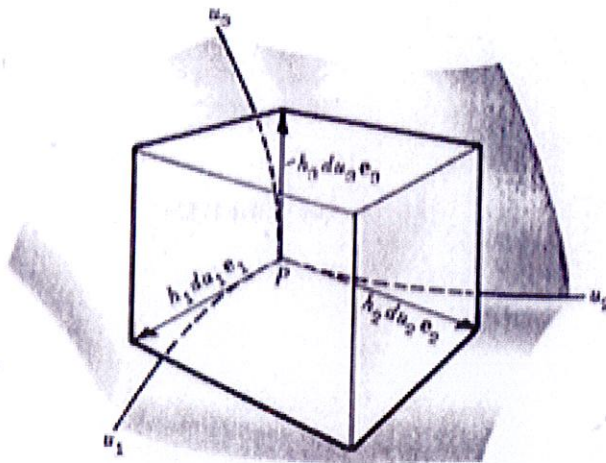
2.1 *Transformation*

Define x, y, z as functions of u_1, u_2, u_3.

$$x = x(u_1, u_2, u_3)$$
$$y = y(u_1, u_2, u_3)$$
$$z = z(u_1, u_2, u_3)$$

Define the position vector $\mathbf{r} = \langle x, y, z \rangle$ and therefore $\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$.

The unit tangent vectors to the intersecting u_1, u_2, and u_3 curves are given by the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ respectively. These vectors form the base vectors of the coordinate system at any point of the surface.

$$e_1 = \frac{\dfrac{\partial r}{\partial u_1}}{\left|\dfrac{\partial r}{\partial u_1}\right|} = \frac{1}{h_1}\frac{\partial r}{\partial u_1}$$

$$e_2 = \frac{\dfrac{\partial r}{\partial u_2}}{\left|\dfrac{\partial r}{\partial u_2}\right|} = \frac{1}{h_2}\frac{\partial r}{\partial u_2}$$

$$e_3 = \frac{\dfrac{\partial r}{\partial u_3}}{\left|\dfrac{\partial r}{\partial u_3}\right|} = \frac{1}{h_3}\frac{\partial r}{\partial u_3}$$

The quantities h_1, h_2, h_3 are called the scaling factors of the system.

Note that $h_1 = \left|\dfrac{\partial r}{\partial u_1}\right| = \sqrt{\left(\dfrac{\partial x}{\partial u_1}\right)^2 + \left(\dfrac{\partial y}{\partial u_1}\right)^2 + \left(\dfrac{\partial z}{\partial u_1}\right)^2}$ is the speed, in terms of time u_1, with which a curve $u_2 =$ constant and $u_3 =$ constant is traced.

From now on we will assume that the basis (e_1, e_2, e_3) forms an orthogonal space. Also assume that the functions x, y, z can be solved in terms of the new coordinates u_1, u_2, u_3. In other words, the system is invertible.

$$x = x(u_1, u_2, u_3) \qquad\qquad u_1 = u_1(x, y, z)$$
$$y = y(u_1, u_2, u_3) \quad\leftrightarrow\quad u_2 = u_2(x, y, z)$$
$$z = z(u_1, u_2, u_3) \qquad\qquad u_3 = u_3(x, y, z)$$

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2.2 Line Element

Let ds represent an element of arc length in the general coordinate system.

$$ds^2 = d\mathbf{r} \cdot d\mathbf{r}$$

$$= \left(\frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 \right) \cdot \left(\frac{\partial \mathbf{r}}{\partial u_1} du_1 + \frac{\partial \mathbf{r}}{\partial u_2} du_2 + \frac{\partial \mathbf{r}}{\partial u_3} du_3 \right)$$

$$= (\mathbf{e}_1 h_1 du_1 + \mathbf{e}_2 h_2 du_2 + \mathbf{e}_3 h_3 du_3) \cdot (\mathbf{e}_1 h_1 du_1 + \mathbf{e}_2 h_2 du_2 + \mathbf{e}_3 h_3 du_3)$$

$$= (\mathbf{e}_1 \cdot \mathbf{e}_1)(h_1 du_1)^2 + (\mathbf{e}_2 \cdot \mathbf{e}_2)(h_2 du_2)^2 + (\mathbf{e}_3 \cdot \mathbf{e}_3)(h_3 du_3)^2$$

By assuming that the base vectors are orthogonal,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then

$$ds^2 = (\mathbf{e}_1 \cdot \mathbf{e}_1)(h_1 du_1)^2 + (\mathbf{e}_2 \cdot \mathbf{e}_2)(h_2 du_2)^2 + (\mathbf{e}_3 \cdot \mathbf{e}_3)(h_3 du_3)^2$$

$$= h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

2.3 Volume Element

Recall that the area of a parallelepiped with vector sides $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ is $V = |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|$.

We see that for an increment of volume, the parallelepiped formed has sides formed from the tangent vectors in each base direction, namely $d\mathbf{r}_1$, $d\mathbf{r}_2$, $d\mathbf{r}_3$.

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u_1} du_1 \cdot \left(\frac{\partial \mathbf{r}}{\partial u_2} du_2 \times \frac{\partial \mathbf{r}}{\partial u_3} du_3 \right) \right|$$

We can factor out constants.

$$dV = \left| \frac{\partial \mathbf{r}}{\partial u_1} \cdot \left(\frac{\partial \mathbf{r}}{\partial u_2} \times \frac{\partial \mathbf{r}}{\partial u_3} \right) \right| du_1 du_2 du_3$$

$$= \left| h_1 \mathbf{e}_1 \cdot (h_2 \mathbf{e}_2 \times h_3 \mathbf{e}_3) \right| du_1 du_2 du_3$$

$$= h_1 h_2 h_3 \left| \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) \right| du_1 du_2 du_3$$

Note that $(\mathbf{e}_2 \times \mathbf{e}_3) = \mathbf{e}_1$ and $\mathbf{e}_1 \cdot \mathbf{e}_1 = 1$ by orthogonality.

$$dV = h_1 h_2 h_3 \, du_1 du_2 du_3$$

2.4 Gradient

The gradient (∇F) measures the rate of change of a scalar field $F(u_1, u_2, u_3)$ at a point, and is expressed in terms of a vector. The direction of the vector is the direction of the greatest rate of increase of F, and the magnitude is the value of the maximum rate of increase [8].

For any scalar function F, we can express the gradient of F in orthogonal curvilinear coordinates (u_1, u_2, u_3) as a linear combination of the base vectors $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

(1) $\nabla F(u_1, u_2, u_3) = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3$

Since

(2) $dr = \dfrac{\partial \mathbf{r}}{\partial u_1} du_1 + \dfrac{\partial \mathbf{r}}{\partial u_2} du_2 + \dfrac{\partial \mathbf{r}}{\partial u_3} du_3 = \mathbf{e}_1 h_1 du_1 + \mathbf{e}_2 h_2 du_2 + \mathbf{e}_3 h_3 du_3$

The differential of a scalar function is defined as $dF = \nabla F \cdot dr$. Thus, using equations (1) and (2),

(3) $\quad dF = (F_1, F_2, F_3) \bullet (h_1 du_1, h_2 du_2, h_3 du_3)$
$\quad\quad\quad = F_1 h_1 du_1 + F_2 h_2 du_2 + F_3 h_3 du_3$

But

(4) $\quad dF = \dfrac{\partial F}{\partial u_1} du_1 + \dfrac{\partial F}{\partial u_2} du_2 + \dfrac{\partial F}{\partial u_3} du_3$

Equating (3) and (4),

$$F_1 = \frac{1}{h_1} \frac{\partial F}{\partial u_1} \, , \quad F_2 = \frac{1}{h_2} \frac{\partial F}{\partial u_2} \, , \quad F_3 = \frac{1}{h_3} \frac{\partial F}{\partial u_3}$$

Then substituting into equation (1), the gradient of the function F is defined as

$$\nabla F = \frac{1}{h_1} \frac{\partial F}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial F}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial F}{\partial u_3} \mathbf{e}_3$$

This indicates the del operator equivalence

$$\nabla \equiv \left(\frac{1}{h_1} \frac{\partial}{\partial u_1} \, , \ \frac{1}{h_2} \frac{\partial}{\partial u_2} \, , \ \frac{1}{h_3} \frac{\partial}{\partial u_3} \right)$$
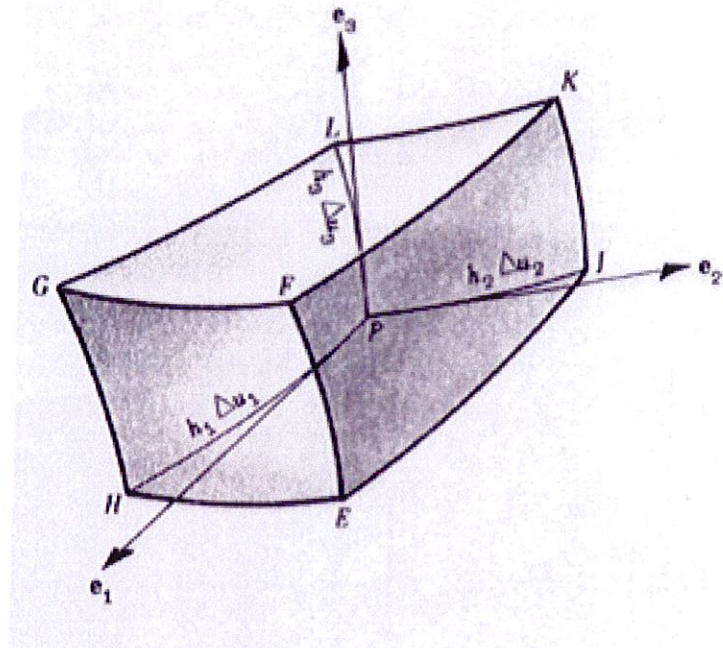
2.5 Divergence

The divergence is equal to the rate of increase of the lines of flow per volume [8].

$$\text{div } \mathbf{A} = \lim_{V \to 0} \frac{\oint \mathbf{A} \cdot d\mathbf{s}}{V}$$

It can also be described as the rate at which "density" exits a given region of space. By measuring the net flux of content passing through a surface surrounding the region of space, it is immediately possible to say how the density of the interior has changed [4].

Let $\mathbf{A} = A_1 \mathbf{e}_1 + A_2 \mathbf{e}_2 + A_3 \mathbf{e}_3$ and let $\mathbf{n}$ be the outward drawn unit normal to the surface ΔS of ΔV.

Notice that P in the diagram is a point on the surface. The procedure to derive the divergence at a point P on the surface is to consider the outward flux of each of the six faces of the volume element ΔV having edges $h_1 \Delta u_1$, $h_2 \Delta u_2$, $h_3 \Delta u_3$. The outward flux $= \iint\limits_{S_i} (\mathbf{A} \cdot \mathbf{ds})$, where $\mathbf{ds}$ is the vector increment of surface area.

$$\mathbf{ds} = (h_i \Delta u_i \mathbf{e}_i \times h_j \Delta u_j \mathbf{e}_j)$$
$$= h_i h_j (\mathbf{e}_i \times \mathbf{e}_j) \Delta u_i \Delta u_j$$

Thus $\mathbf{ds} = h_i h_j \Delta u_i \Delta u_j \mathbf{e}_k$ where $\mathbf{e}_k = \mathbf{n}$, and it can be substituted into the flux integral.

$$\iint\limits_{S_i} (\mathbf{A} \cdot \mathbf{ds}) = \iint\limits_{S_i} (\mathbf{A} \cdot \mathbf{n}) \, ds$$

Extending the mean value theorem for integrals, we can express the flux as $(\mathbf{A} \cdot \mathbf{n})$ at a point on the surface S_i multiplied by the area of S_i .

(5)
$$\iint_{S_i} (\mathbf{A} \cdot d\mathbf{s}) = [(\mathbf{A} \cdot \mathbf{n}) \text{ at a point}] \iint_{S_i} ds = [(\mathbf{A} \cdot \mathbf{n}) \text{ at a point}]*[\text{area of } S_i]$$

By assuming continuity, $(\mathbf{A} \cdot \mathbf{n})$ at any point on the surface approximates $(\mathbf{A} \cdot \mathbf{n})$ at P. Thus, when calculating the divergence at a point P, this approximation is sufficient.

We will use equation (5) to derive the flux on each of the six faces, simplifying the task by taking advantage of symmetry.

Consider the surface PLKJ. The outward unit normal is $\mathbf{n} = -\mathbf{e}_1$. Let surface PLKJ = S_1 .

Then

(6)
$$\iint_{S_1} (\mathbf{A} \cdot d\mathbf{s}) = \iint_{S_1} (\mathbf{A} \cdot -\mathbf{e}_1) \, h_2 h_3 du_2 du_3$$
$$= -A_1(u_1, u_2, u_3) \, h_2 h_3 \Delta u_2 \Delta u_3$$

Now consider the surface EFGH = S_2 . The outward unit normal is $\mathbf{n} = \mathbf{e}_1$.

(7)
$$\iint_{S_2} (\mathbf{A} \cdot d\mathbf{s}) = \iint_{S_2} (\mathbf{A} \cdot \mathbf{e}_1) \, h_2 h_3 du_2 du_3$$
$$= \iint_{S_2} \underbrace{A_1(u_1 + h_1 \Delta u_1, u_2, u_3) h_2 h_3 du_2 du_3}$$

We must expand the underlined function using Taylor's Expansion to obtain a linear approximation of the function $A_1(u_1 + h_1 \Delta u_1, u_2, u_3) \, h_2 h_3$. Taylor's expansion is given by

$$\mathbf{f}(x + \mathbf{p}) = \mathbf{f}(x) + D_{\mathbf{p}} \mathbf{f}(x) \|\mathbf{p}\|$$
$$= \mathbf{f}(x) + \nabla \mathbf{f}(x) \cdot \frac{\mathbf{p}}{\|\mathbf{p}\|} \|\mathbf{p}\|$$
$$= \mathbf{f}(x) + \nabla \mathbf{f}(x) \cdot \mathbf{p}$$

Thus, with $\mathbf{p} = h_1 \Delta u_1 \mathbf{e}_1$, Taylor's expansion of the function results in the approximation

$$A_1(u_1+h_1\Delta u_1, u_2, u_3)\, h_2 h_3 \approx$$
$$A_1(u_1, u_2, u_3)\, h_2 h_3 + \nabla(A_1(u_1, u_2, u_3)\, h_2 h_3) \cdot h_1 \Delta u_1 \mathbf{e}_1$$

Recall

$$\nabla \equiv \left(\frac{1}{h_1}\frac{\partial \mathbf{F}}{\partial u_1}\,,\, \frac{1}{h_2}\frac{\partial \mathbf{F}}{\partial u_2}\,,\, \frac{1}{h_3}\frac{\partial \mathbf{F}}{\partial u_3} \right)$$

So

$$A_1(u_1+h_1\Delta u_1, u_2, u_3)\, h_2 h_3 \approx$$
$$A_1(u_1, u_2, u_3)\, h_2 h_3 + \frac{\partial}{\partial u_1}(A_1(u_1, u_2, u_3)\, h_2 h_3 \Delta u_1)$$

Returning to the surface EFGH and equation (7)

$$\iint\limits_{S_2} (\mathbf{A} \cdot d\mathbf{s}) = \iint\limits_{S_2} A_1(u_1+h_1\Delta u_1, u_2, u_3)\, h_2 h_3\, du_2 du_3$$

$$= \left(A_1(u_1, u_2, u_3)\, h_2 h_3 + \frac{\partial}{\partial u_1}(A_1(u_1, u_2, u_3)\, h_2 h_3 \Delta u_1) \right) \Delta u_2 \Delta u_3$$

$$= A_1(u_1, u_2, u_3)\, h_2 h_3 \Delta u_2 \Delta u_3 + \frac{\partial}{\partial u_1}(A_1(u_1, u_2, u_3)\, h_2 h_3)\Delta u_1 \Delta u_2 \Delta u_3$$

The contribution from surfaces PLKJ (S_1) and EFGH (S_2) is

$$\iint\limits_{S_1} (\mathbf{A} \cdot d\mathbf{s}) + \iint\limits_{S_2} (\mathbf{A} \cdot d\mathbf{s}) = \frac{\partial}{\partial u_1}(A_1(u_1, u_2, u_3)\, h_2 h_3)\Delta u_1 \Delta u_2 \Delta u_3$$

By similar arguments, we can calculate the contribution from the remaining four surfaces:

$$\iint\limits_{FGLK} (\mathbf{A} \cdot d\mathbf{s}) + \iint\limits_{EHPJ} (\mathbf{A} \cdot d\mathbf{s}) = \frac{\partial}{\partial u_3}(A_3(u_1, u_2, u_3)\, h_1 h_2)\Delta u_1 \Delta u_2 \Delta u_3$$

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$$\iint\limits_{GHLP} (\mathbf{A} \cdot d\mathbf{s}) + \iint\limits_{EFKJ} (\mathbf{A} \cdot d\mathbf{s}) = \frac{\partial}{\partial u_2}(A_2(u_1, u_2, u_3) h_1 h_3)\Delta u_1 \Delta u_2 \Delta u_3$$

The total contribution from the six faces of ΔV is

$$\left(\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_1 h_3) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2) \right) \Delta u_1 \Delta u_2 \Delta u_3$$

Dividing this by the volume $h_1 h_2 h_3 \Delta u_1 \Delta u_2 \Delta u_3$ and taking the limit as the volume approaches zero

$$\text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_1 h_3) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2) \right)$$

2.6 Curl

The curl of a vector $\mathbf{A}$ can be physically interpreted as the amount of "rotation" or angular momentum of the contents of a given region of space. The curl is defined as the vector field having magnitude equal to the maximum "circulation" at each point and to be oriented perpendicularly to this plane of circulation for each point. To calculate the curl, we find the limiting value of circulation per unit area A.

$$(\nabla \times \mathbf{F}) \cdot \mathbf{n} = \lim_{A \to 0} \frac{\oint_C \mathbf{F} \cdot d\mathbf{s}}{A}$$

To calculate the curl in orthogonal curvilinear coordinates, we will begin by calculating (curl $\mathbf{F}$) $\cdot$ $\mathbf{e}_1$, the first component of the curl. Consider the surface S_1 normal to $\mathbf{e}_1$ at P, as shown in the figure.
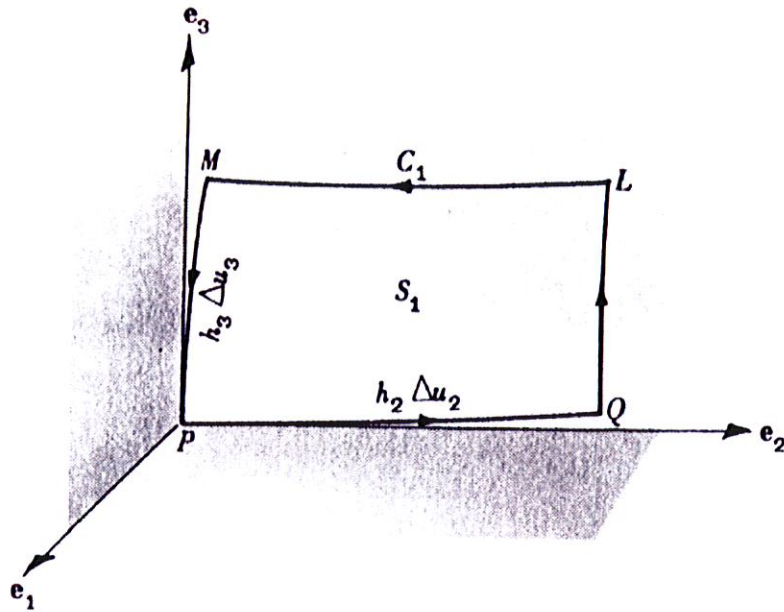
Let $\mathbf{F} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$ and denote the boundary of S_1 by C_1, we have

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{PQ} \mathbf{F} \cdot d\mathbf{r} + \int_{QL} \mathbf{F} \cdot d\mathbf{r} + \int_{LM} \mathbf{F} \cdot d\mathbf{r} + \int_{MP} \mathbf{F} \cdot d\mathbf{r}$$

Again, we will utilize an extended mean value theorem for integrals and assume continuity, therefore the following approximation holds

$$(1) \qquad \int_{PQ} \mathbf{F} \cdot d\mathbf{r} = (\mathbf{F} \text{ at a point}) \cdot (h_2\Delta u_2\mathbf{e}_2)$$
$$= (F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3) \cdot (h_2\Delta u_2\mathbf{e}_2)$$
$$= F_2 h_2\Delta u_2$$

By Taylor's Expansion,

$$\int_{ML} \mathbf{F} \cdot d\mathbf{r} = F_2 h_2\Delta u_2 + \frac{\partial}{\partial u_3}(F_2 h_2\Delta u_2)\Delta u_3$$

or

(2) $\qquad \int_{LM} \mathbf{F} \cdot d\mathbf{r} = -F_2 h_2 \Delta u_2 - \frac{\partial}{\partial u_3}(F_2 h_2 \Delta u_2)\Delta u_3$

Similarly,

$$\int_{PM} \mathbf{F} \cdot d\mathbf{r} = (\mathbf{F} \text{ at } P) \cdot (h_3 \Delta u_3 \mathbf{e}_3) = F_3 h_3 \Delta u_3$$

or

(3) $\qquad \int_{MP} \mathbf{F} \cdot d\mathbf{r} = -F_3 h_3 \Delta u_3$

and

(4) $\qquad \int_{QL} \mathbf{F} \cdot d\mathbf{r} = F_3 h_3 \Delta u_3 + \frac{\partial}{\partial u_2}(F_3 h_3 \Delta u_3)\Delta u_2$

Adding (1), (2), (3) and (4) we have

$$\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = \frac{\partial}{\partial u_2}(F_3 h_3 \Delta u_3)\Delta u_2 - \frac{\partial}{\partial u_3}(F_2 h_2 \Delta u_2)\Delta u_3$$

$$= \left(\frac{\partial}{\partial u_2}(F_3 h_3) - \frac{\partial}{\partial u_3}(F_2 h_2) \right) \Delta u_2 \Delta u_3$$

Dividing by the area of S_1 equal to $h_2 h_3 \Delta u_2 \Delta u_3$ and taking the limit as $A = h_2 h_3 \Delta u_2 \Delta u_3$ approaches zero,

$$(\text{curl } \mathbf{F}) \cdot \mathbf{e}_1 = \frac{1}{h_2 h_3}\left(\frac{\partial}{\partial u_2}(F_3 h_3) - \frac{\partial}{\partial u_3}(F_2 h_2) \right)$$

Similarly, by choosing area S_2 and S_3 perpendicular to $\mathbf{e}_2$ and $\mathbf{e}_3$ at P respectively, we find $(\text{curl } \mathbf{F}) \cdot \mathbf{e}_2$ and $(\text{curl } \mathbf{F}) \cdot \mathbf{e}_3$, the second and third components of the curl vector.

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$$\text{curl } \mathbf{F} = \frac{\mathbf{e}_1}{h_2 h_3}\left(\frac{\partial}{\partial u_2}(F_3 h_3) - \frac{\partial}{\partial u_3}(F_2 h_2)\right)$$

$$+ \frac{\mathbf{e}_2}{h_3 h_1}\left(\frac{\partial}{\partial u_3}(F_1 h_1) - \frac{\partial}{\partial u_1}(F_3 h_3)\right) + \frac{\mathbf{e}_3}{h_1 h_2}\left(\frac{\partial}{\partial u_1}(F_2 h_2) - \frac{\partial}{\partial u_2}(F_1 h_1)\right)$$

or

$$\text{curl } \mathbf{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

2.7 Laplacian

The Laplacian of a scalar function is defined to be the divergence of the gradient. We make use of the divergence equation derived above.

$$\nabla^2 \mathbf{A} = \nabla \cdot \left\langle \frac{1}{h_1}\frac{\partial \mathbf{A}}{\partial u_1}, \frac{1}{h_2}\frac{\partial \mathbf{A}}{\partial u_2}, \frac{1}{h_3}\frac{\partial \mathbf{A}}{\partial u_3} \right\rangle$$

$$= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{1}{h_1}\frac{\partial \mathbf{A}}{\partial u_1}h_2 h_3\right) + \frac{\partial}{\partial u_2}\left(\frac{1}{h_2}\frac{\partial \mathbf{A}}{\partial u_2}h_1 h_3\right) + \frac{\partial}{\partial u_3}\left(\frac{1}{h_3}\frac{\partial \mathbf{A}}{\partial u_3}h_1 h_2\right)\right)$$

$$= \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \mathbf{A}}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \mathbf{A}}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial \mathbf{A}}{\partial u_3}\right)\right)$$

3. APPLICATIONS

3.1 *Ring Coordinates*

Kellog presents a problem in his book, *Foundations of Potential Theory*, asking the reader to verify the line element ds² and the Laplacian ∇²U, given the following equations.

$$x = r\cos\varphi \,, \quad y = r\sin\varphi \,, \quad z = \frac{\sin\mu}{\cosh\lambda + \cos\mu} \,,$$

$$\text{where } r = \frac{\sinh\lambda}{\cosh\lambda + \cos\mu}$$

Verify

$$ds^2 = r^2\left[\frac{d\lambda^2 + d\mu^2}{\sinh^2\lambda} + d\varphi^2\right]$$

$$\nabla^2 U = \frac{\sinh^2\lambda}{r^3}\left[\frac{\partial}{\partial\lambda}\left(r\frac{\partial U}{\partial\lambda}\right) + \frac{\partial}{\partial\mu}\left(r\frac{\partial U}{\partial\mu}\right) + \frac{r}{\sinh^2\lambda}\frac{\partial^2 U}{\partial\varphi^2}\right]$$

I chose this problem to confirm the validity of my general results derived for curvilinear coordinates in the previous section.

The first task when approaching any curvilinear coordinate system is to determine the base vectors and scaling factors for the system. Let $\mathbf{F}(\lambda, \mu, \varphi) = \langle x, y, z\rangle$ be the position vector where $x = x(\lambda, \mu, \varphi)$, $y = y(\lambda, \mu, \varphi)$, and $z = z(\lambda, \mu, \varphi)$.

$$(1) \quad \mathbf{e}_\lambda = \frac{\dfrac{\partial\mathbf{F}}{\partial\lambda}}{\left|\dfrac{\partial\mathbf{F}}{\partial\lambda}\right|} = \frac{\left\langle \cos\varphi\dfrac{\partial r}{\partial\lambda} \,, \, \sin\phi\dfrac{\partial r}{\partial\lambda} \,, \dfrac{\sinh\lambda \, \sin\mu}{(\cosh\lambda + \cos\mu)^2}\right\rangle}{\sqrt{\left(\dfrac{\partial r}{\partial\lambda}\right)^2(\cos^2\phi + \sin^2\phi) + \left(\dfrac{\sinh\lambda \, \sin\mu}{(\cosh\lambda + \cos\mu)^2}\right)^2}}$$

and so

$$(2) \quad h_\lambda = \sqrt{\left(\frac{\partial r}{\partial \lambda}\right)^2 + \left(\frac{\sinh\lambda \, \sin\mu}{(\cosh\lambda + \cos\mu)^2}\right)^2}$$

We must find $\dfrac{\partial r}{\partial \lambda}$ to complete h_λ .

$$(3) \quad \frac{\partial r}{\partial \lambda} = \frac{(\cosh\lambda + \cos\mu)\cosh\lambda - \sinh\lambda(\sinh\lambda)}{(\cosh\lambda + \cos\mu)^2}$$

therefore, substituting (3) into (2)

$$(4) \quad h_\lambda = \sqrt{\frac{[(\cosh^2\lambda - \sinh^2\lambda) + \cosh\lambda\cos\mu]^2}{(\cosh\lambda + \cos\mu)^4} + \frac{\sinh^2\lambda \, \sin^2\mu}{(\cosh\lambda + \cos\mu)^4}}$$

$$= \frac{1}{(\cosh\lambda + \cos\mu)^2}\sqrt{(1 + \cosh\lambda\cos\mu)^2 + \sinh^2\lambda \, \sin^2\mu}$$

Let

$$\alpha = \frac{1}{(\cosh\lambda + \cos\mu)^2}$$

then

$$(5) \quad h_\lambda = \alpha\sqrt{1 + 2\cosh\lambda\cos\mu + \cosh^2\lambda\cos^2\mu + \sinh^2\lambda \, \sin^2\mu}$$

$$= \alpha\sqrt{1 + 2\cosh\lambda\cos\mu + \cos^2\mu(1 + \sinh^2\lambda) + \sinh^2\lambda \, (1 - \cos^2\mu)}$$

$$= \alpha\sqrt{1 + 2\cosh\lambda\cos\mu + \cos^2\mu + \sinh^2\lambda}$$

$$= \alpha\sqrt{1 + 2\cosh\lambda\cos\mu + \cos^2\mu + (\cosh^2\lambda - 1)}$$

$$= \alpha\sqrt{(\cosh\lambda + \cos\mu)^2}$$

Substituting for α and simplifying, we see that

(6) $\qquad h_\lambda = \dfrac{1}{\cosh\lambda + \cos\mu}$

Note that the general formulas for the line element and the Laplacian in curvilinear coordinates depend only on the scaling factors h_1, h_2, h_3. We must calculate $\mathbf{e}_\mu$ and $\mathbf{e}_\phi$ to obtain h_μ and h_φ.

(7) $\qquad \mathbf{e}_\mu = \dfrac{\dfrac{\partial \mathbf{F}}{\partial\mu}}{\left|\dfrac{\partial \mathbf{F}}{\partial\mu}\right|} = \dfrac{\left\langle \cos\varphi\dfrac{\partial r}{\partial\mu} \ , \ \sin\phi\dfrac{\partial r}{\partial\mu} \ , \ \dfrac{\cosh\lambda\cos\mu + 1}{(\cosh\lambda + \cos\mu)^2} \right\rangle}{\sqrt{\left(\dfrac{\partial r}{\partial\mu}\right)^2 (\cos^2\phi + \sin^2\phi) + \left(\dfrac{\cosh\lambda\cos\mu + 1}{(\cosh\lambda + \cos\mu)^2}\right)^2}}$

where

(8) $\qquad h_\mu = \sqrt{\left(\dfrac{\partial r}{\partial\mu}\right)^2 + \left(\dfrac{\cosh\lambda\cos\mu + 1}{(\cosh\lambda + \cos\mu)^2}\right)^2}$

Find $\dfrac{\partial r}{\partial\mu}$ to complete h_μ .

(9) $\qquad \dfrac{\partial r}{\partial\mu} = \dfrac{-\sinh\lambda\sin\mu}{(\cosh\lambda + \cos\mu)^2}$

therefore, substituting (9) into (8)

(10) $\qquad h_\mu = \sqrt{\left(\dfrac{-\sinh\lambda\sin\mu}{(\cosh\lambda + \cos\mu)^2}\right)^2 + \left(\dfrac{\cosh\lambda\cos\mu + 1}{(\cosh\lambda + \cos\mu)^2}\right)^2}$

$\qquad\qquad = \dfrac{1}{(\cosh\lambda + \cos\mu)^2}\sqrt{(1 + \cosh\lambda\cos\mu)^2 + \sinh^2\lambda\ \sin^2\mu}$

Notice that (10) is equivalent to the last equation in (4). Thus, the

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same algebra applies.

$$(11) \quad h_{\mu} = h_{\lambda} = \frac{1}{\cosh\lambda + \cos\mu}$$

Repeating the process for $\mathbf{e}_{\phi}$,

$$(12) \quad e_{\phi} = \frac{\dfrac{\partial \mathbf{F}}{\partial \varphi}}{\left| \dfrac{\partial \mathbf{F}}{\partial \varphi} \right|} = \frac{\langle -r\sin\varphi, \ r\cos\varphi , \ 0 \rangle}{\sqrt{r^2(\cos^2\phi + \sin^2\phi)}}$$

So

$$(13) \quad h_{\phi} = r$$

Now we are able to substitute into our general formulas for the line element and Laplacian.

Recall

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

The line element in Ring Coordinates is

$$(14) \quad ds^2 = \left(\frac{1}{\cosh\lambda + \cos\mu} \right)^2 d\lambda^2 + \left(\frac{1}{\cosh\lambda + \cos\mu} \right)^2 d\mu^2 + r^2 d\varphi^2$$

and given $r = \dfrac{\sinh\lambda}{\cosh\lambda + \cos\mu}$, we have verified that

$$ds^2 = r^2 \left(\frac{d\lambda^2 + d\mu^2}{\sinh^2\lambda} + d\varphi^2 \right)$$

Also recall

$$\nabla^2 U = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial U}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial U}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial U}{\partial u_3}\right)\right)$$

The Laplacian in Ring Coordinates is

$$(15) \qquad \nabla^2 U = \frac{(\cosh\lambda + \cos\mu)^2}{r}\left(\frac{\partial}{\partial\lambda}\left(r\frac{\partial U}{\partial\lambda}\right) + \frac{\partial}{\partial\mu}\left(r\frac{\partial U}{\partial\mu}\right) + \right.$$
$$\left. \frac{\partial}{\partial\varphi}\left(\frac{1}{r(\cosh\lambda + \cos\mu)^2}\frac{\partial U}{\partial\varphi}\right)\right)$$

Since

$$r = \frac{\sinh\lambda}{\cosh\lambda + \cos\mu}$$

we get

$$\cosh\lambda + \cos\mu = \frac{\sinh\lambda}{r}$$

and then

$$(16) \qquad \nabla^2 U = \frac{\sinh^2\lambda}{r^3}\left(\frac{\partial}{\partial\lambda}\left(r\frac{\partial U}{\partial\lambda}\right) + \frac{\partial}{\partial\mu}\left(r\frac{\partial U}{\partial\mu}\right) + \frac{1}{r(\cosh\lambda + \cos\mu)^2}\frac{\partial^2 U}{\partial\varphi^2}\right)$$

and

$$\frac{1}{r(\cosh\lambda + \cos\mu)^2} = \frac{1}{r\dfrac{\sinh^2\lambda}{r^2}} = \frac{r}{\sinh^2\lambda} \ .$$

Therefore, we have verified the Laplacian in Ring Coordinates.

$$(17) \qquad \nabla^2 U = \frac{\sinh^2\lambda}{r^3}\left(\frac{\partial}{\partial\lambda}\left(r\frac{\partial U}{\partial\lambda}\right) + \frac{\partial}{\partial\mu}\left(r\frac{\partial U}{\partial\mu}\right) + \frac{r}{\sinh^2\lambda}\frac{\partial^2 U}{\partial\varphi^2} \right)$$

3.2 *The Helmholtz Differential Equation*

Curvilinear coordinates are commonly used in solving elliptic partial differential equations, especially in situations where the process is simplified when converted from Cartesian coordinates to a more suitable coordinated system. Elliptic partial differential equations have applications in almost all areas of mathematics, from harmonic analysis to geometry to Lie theory, as well as numerous applications in physics [9].

The Helmholtz differential equation

$$\nabla^2 F + k^2 F = 0$$

is one example of an elliptic partial differential equation. When k = 0, the Helmholtz differential equation reduces to Laplace's equation.

The Helmholtz differential equation can be solved by separation of variables in eleven coordinate systems, ten of which are particular cases of the confocal ellipsoidal system. As part of my research, I chose to demonstrate separation of variables for the Helmholtz equation using parabolic coordinates and elliptic cylindrical coordinates. The approach of separating variables does not solve the equation, but makes the solution "easily" obtained with techniques for solving ordinary differential equations.

3.2.1 *Parabolic Coordinates*

Our goal is to separate variables in the Helmholtz differential equation.

$$\nabla^2 F + k^2 F = 0$$

by substituting the Laplacian in parabolic coordinates into the equation for $\nabla^2 F$, where F is a function $F = F(u, v, \theta)$.

Parabolic coordinates are defined as follows:

(1) $x = uv\cos\theta$
       $y = uv\sin\theta$
       $z = \frac{1}{2}(u^2 - v^2)$

Let $\mathbf{r} = (x, y, z)$. We find the base vectors $\mathbf{e}_u$, $\mathbf{e}_v$, $\mathbf{e}_\theta$ and scaling factors h_u, h_v, h_θ in order to determine the Laplacian.

$$\mathbf{e}_u = \frac{\dfrac{\partial \mathbf{r}}{\partial u}}{\left| \dfrac{\partial \mathbf{r}}{\partial u} \right|} = \frac{\langle\, v\cos\theta\, ,\, v\sin\theta,\, u\, \rangle}{\sqrt{v^2 + u^2}} \; , \quad \text{where} \quad h_u = \sqrt{u^2 + v^2}$$

$$\mathbf{e}_v = \frac{\dfrac{\partial \mathbf{r}}{\partial v}}{\left| \dfrac{\partial \mathbf{r}}{\partial v} \right|} = \frac{\langle\, u\cos\theta\, ,\, u\sin\theta,\, v\, \rangle}{\sqrt{u^2 + v^2}} \; , \quad \text{where} \quad h_v = \sqrt{u^2 + v^2}$$

$$\mathbf{e}_\theta = \frac{\dfrac{\partial \mathbf{r}}{\partial \theta}}{\left| \dfrac{\partial \mathbf{r}}{\partial \theta} \right|} = \frac{\langle\, -uv\sin\theta\, ,\, uv\cos\theta,\, 0\, \rangle}{\sqrt{u^2 v^2}} \; , \quad \text{where} \quad h_\theta = uv$$

Now we are able to find the Laplacian in parabolic coordinates.

$$(2) \quad \nabla^2 F = \frac{1}{uv(u^2+v^2)}\left(\frac{\partial}{\partial u}\left(uv\frac{\partial F}{\partial u}\right) + \frac{\partial}{\partial v}\left(uv\frac{\partial F}{\partial v}\right) + \frac{\partial}{\partial \theta}\left(\frac{(u^2+v^2)}{uv}\frac{\partial F}{\partial \theta}\right)\right)$$

Using the product rule of differentiation, our result is

$$(3) \quad \nabla^2 F = \frac{1}{uv(u^2+v^2)}\left(v\frac{\partial F}{\partial u} + uv\frac{\partial^2 F}{\partial u^2} + u\frac{\partial F}{\partial v} + uv\frac{\partial^2 F}{\partial v^2} + \frac{(u^2+v^2)}{uv}\frac{\partial^2 F}{\partial \theta^2}\right)$$

Simplifying equation (3) gives,

$$(4) \quad \nabla^2 F = \frac{1}{(u^2+v^2)}\left(\frac{1}{u}\frac{\partial F}{\partial u} + \frac{\partial^2 F}{\partial u^2} + \frac{1}{v}\frac{\partial F}{\partial v} + \frac{\partial^2 F}{\partial v^2}\right) + \frac{1}{u^2 v^2}\frac{\partial^2 F}{\partial \theta^2}$$

So the Helmholtz equation in parabolic coordinates is

$$(5) \quad \frac{1}{(u^2+v^2)}\left(\frac{1}{u}\frac{\partial F}{\partial u} + \frac{\partial^2 F}{\partial u^2} + \frac{1}{v}\frac{\partial F}{\partial v} + \frac{\partial^2 F}{\partial v^2}\right) + \frac{1}{u^2 v^2}\frac{\partial^2 F}{\partial \theta^2} + k^2 F = 0$$

Attempt separation of variables by letting

$$(6) \quad F(u, v, \theta) = U(u)V(v)\Theta(\theta)$$

be a solution to the differential equation. Then the Helmholtz equation (5) becomes

(7)

$$\frac{1}{(u^2+v^2)}\left(V\Theta\left(\frac{1}{u}\frac{dU}{du} + \frac{d^2 U}{du^2}\right) + U\Theta\left(\frac{1}{v}\frac{dV}{dv} + \frac{d^2 V}{dv^2}\right)\right) + \frac{UV}{u^2 v^2}\frac{d^2\Theta}{d\theta^2} + k^2\, UV\Theta = 0$$

Multiplying equation (7) through by $\dfrac{u^2v^2}{UV\theta}$,

(8)

$$\frac{u^2v^2}{(u^2+v^2)}\left[\frac{1}{U}\left(\frac{1}{u}\frac{dU}{du}+\frac{d^2U}{du^2}\right)+\frac{1}{V}\left(\frac{1}{v}\frac{dV}{dv}+\frac{d^2V}{dv^2}\right)\right]+\frac{1}{\theta}\frac{d^2\theta}{d\theta^2}+k^2u^2v^2=0$$

In equation (8), we have separated the Helmholtz equation into a function G(u, v) and a function H(θ). Consequently, H(θ) must be equal to a constant. More specifically, H(θ) = −G(u, v).

Let $H(\theta)=\dfrac{1}{\theta}\dfrac{d^2\theta}{d\theta^2}=-m^2$, where m is a real.

Substituting into equation (8) and multiplying by $\dfrac{(u^2+v^2)}{u^2v^2}$

(9)

$$\left(\frac{1}{U}\left(\frac{1}{u}\frac{dU}{du}+\frac{d^2U}{du^2}\right)+\frac{1}{V}\left(\frac{1}{v}\frac{dV}{dv}+\frac{d^2V}{dv^2}\right)\right)-m^2\frac{(u^2+v^2)}{u^2v^2}+k^2(u^2+v^2)=0$$

Rewriting $\dfrac{(u^2+v^2)}{u^2v^2}=\left(\dfrac{1}{u^2}+\dfrac{1}{v^2}\right)$ gives,

(10)

$$\left(\frac{1}{U}\left(\frac{1}{u}\frac{dU}{du}+\frac{d^2U}{du^2}\right)+\frac{1}{V}\left(\frac{1}{v}\frac{dV}{dv}+\frac{d^2V}{dv^2}\right)\right)-m^2\left(\frac{1}{u^2}+\frac{1}{v^2}\right)+k^2(u^2+v^2)=0$$

Equation (10) can now be rearranged into two terms, each containing only u or v.

(11)

$$\left(\frac{1}{U}\left(\frac{1}{u}\frac{dU}{du}+\frac{d^2U}{du^2}\right)+k^2u^2-\frac{m^2}{u^2}\right)+\left(\frac{1}{V}\left(\frac{1}{v}\frac{dV}{dv}+\frac{d^2V}{dv^2}\right)+k^2v^2-\frac{m^2}{v^2}\right)=0$$

We have succeeded in separating the Helmholtz differential equation using parabolic coordinates into three independent functions.

3.2.2 *Elliptic Cylindrical Coordinates*

Elliptic cylindrical coordinates are defined as follows:

$$(12) \quad \begin{aligned} x &= a*\cosh(u)\cos(v) \\ y &= a*\sinh(u)\sin(v) \\ z &= z \end{aligned} \qquad \text{where} \qquad \begin{aligned} 0 &\le u < \infty \\ 0 &\le v < 2\pi \\ -\infty &\le z < \infty \end{aligned}$$

In this transformation, the v coordinates are the asymptotic angle of confocal hyperbolic cylinders symmetrical about the x–axis. The u coordinates are confocal elliptic cylinders centered on the origin.

Let $\mathbf{r} = (x, y, z)$. As usual, we will begin by finding the base vectors and scaling factors.

$$\mathbf{e}_u = \frac{\dfrac{\partial \mathbf{r}}{\partial u}}{\left| \dfrac{\partial \mathbf{r}}{\partial u} \right|} = \frac{\langle\, a*\sinh(u)\cos(v)\,,\, a*\cosh(u)\sin(v),\, 0\,\rangle}{\sqrt{a^2(\sinh^2(u)\cos^2(v) + \cosh^2(u)\sin^2(v))}}$$

and so,

$$h_u = a\sqrt{\sinh^2(u) + \sin^2(v)}$$

Note, the algebra to reduce h_u is shown below. Recall that $\cosh^2 x - \sinh^2 x = 1$.

$$\begin{aligned} \sinh^2(u)\cos^2(v) &+ \cosh^2(u)\sin^2(v) \\ &= \sinh^2(u)(1 - \sin^2(v)) + (1 + \sinh^2(u))\sin^2(v) \\ &= \sinh^2(u) - \sinh^2(u)\sin^2(v) + \sin^2(v) + \sinh^2(u)\sin^2(v) \\ &= \sinh^2(u) + \sin^2(v) \end{aligned}$$

Computing the base vector $\mathbf{e}_v$,

$$\mathbf{e}_v = \frac{\dfrac{\partial \mathbf{r}}{\partial v}}{\left| \dfrac{\partial \mathbf{r}}{\partial v} \right|} = \frac{\langle\, -a*\cosh(u)\sin(v)\, ,\, a*\sinh(u)\cos(v),\, 0\, \rangle}{\sqrt{a^2(\cosh^2(u)\sin^2(v) + \sinh^2(u)\cos^2(v))}}$$

Notice

$$h_v = h_u = a\sqrt{\sinh^2(u) + \sin^2(v)}$$

Also

$$\mathbf{e}_z = \frac{\dfrac{\partial \mathbf{r}}{\partial z}}{\left| \dfrac{\partial \mathbf{r}}{\partial z} \right|} = \frac{\langle\, 0,\, 0,\, 1\, \rangle}{1}$$

and

$$h_z = 1$$

The Laplacian in elliptic cylindrical coordinates is

(13)

$$\nabla^2 F = \frac{1}{a^2(\sinh^2(u) + \sin^2(v))}\left(\frac{\partial}{\partial u}\left(\frac{\partial F}{\partial u} \right) + \frac{\partial}{\partial v}\left(\frac{\partial F}{\partial v} \right) + \frac{\partial}{\partial z}\left(\frac{\sinh^2(u) + \sin^2(v)}{1} \frac{\partial F}{\partial z} \right) \right)$$

Simplifying,

(14) $$\nabla^2 F = \frac{1}{a^2(\sinh^2(u) + \sin^2(v))}\left(\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2} \right) + \frac{\partial^2 F}{\partial z^2}$$

We can now substitute the Laplacian into the Helmholtz differential equation, $\nabla^2 F + k^2 F = 0$.

(15)
$$\frac{1}{a^2(\sinh^2(u) + \sin^2(v)))}\left(\frac{\partial^2 F}{\partial u^2} + \frac{\partial^2 F}{\partial v^2}\right) + \frac{\partial^2 F}{\partial z^2} + k^2 F = 0$$

Let

(16)
$$F(u, v, z) = U(u)V(v)Z(z)$$

Assuming $F(u, v, z)$ is a solution of the Helmholtz equation, (15) becomes

(17)
$$\frac{Z}{a^2(\sinh^2(u) + \sin^2(v))}\left(V\frac{d^2U}{du^2} + U\frac{d^2V}{dv^2}\right) + UV\frac{d^2Z}{dz^2} + k^2 UVZ = 0$$

Divide by UVZ

(18)
$$\frac{1}{a^2(\sinh^2(u) + \sin^2(v))}\left(\frac{1}{U}\frac{d^2U}{du^2} + \frac{1}{V}\frac{d^2V}{dv^2}\right) + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0$$

The equation is now separated into a function $G(u, v)$ and a function $H(z)$. We can conclude that $H(z)$ is equal to a constant, $H(z) = -G(u, v)$.

Let $H(z) = \dfrac{1}{Z}\dfrac{d^2Z}{dz^2} = -(k^2 + m^2)$, where m is a real.

Substituting into equation (18) and multiplying through by $a^2(\sinh^2(u) + \sin^2(v))$ gives

(19)
$$\left(\frac{1}{U}\frac{d^2U}{du^2} + \frac{1}{V}\frac{d^2V}{dv^2}\right) = m^2 a^2(\sinh^2(u) + \sin^2(v))$$

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Thus

$$(20) \quad \left[\frac{1}{U} \frac{d^2 U}{du^2} - m^2 a^2 \sinh^2(u) \right] = \left[m^2 a^2 \sin^2(v) - \frac{1}{V} \frac{d^2 V}{dv^2} \right]$$

We have shown that the Helmholtz differential equation is separable in elliptic cylindrical coordinates.

4. CONCLUSIONS

Along with many mathematical processes, the utility of curvilinear coordinates in solving partial differential equations has long been replaced by technology's high speed numerical analysis. Its physical interpretation, nonetheless, cannot be overlooked. Gauss had a vision of freeing a world confined by a rectangular grid. His forward thinking and the creation of curvilinear coordinates helped the intellectual community to see mathematics from a different frame of reference. Einstein also recognized the significance of curvilinear coordinates in representing the physical world.

By generally deriving properties of a surface in curvilinear coordinates, we are able to attain information about the surface simply by knowing the transformation from Cartesian coordinates to a new system. The importance lies in the base vectors of the coordinate system at each point and their corresponding scaling factors. This data allows us to easily compute the volume, gradient, divergence, curl, and Laplacian.

Orthogonal curvilinear coordinates are usually applied to physical problems. Mathematicians and physicists find these transformations from the Cartesian coordinate system especially useful when solving partial differential equations that model physical attributes of a curved surfaces. The Helmholtz differential equation, for example, models resonance, the acoustic effect that boosts volume in a drum or bottle shaped object. The technique of separation of variables can be used in solving Helmholtz's equation in 11 different orthogonal curvilinear coordinate systems, including the parabolic and the elliptic cylindrical coordinate systems.

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APPENDIX A: CYLINDRICAL COORDINATES

The derivation of the following mathematical quantities utilizes the general derivation discussed in Section 2. It is shown here as a familiar resource.

Transformation:

$$x = r*\cos\theta$$
$$y = r*\sin\theta$$
$$z = z$$

and

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$z = z$$

Let $\mathbf{r} = (x, y, z)$.

Base Vectors and Scaling Factors:

$$\mathbf{e}_r = \frac{\dfrac{\partial \mathbf{r}}{\partial r}}{\left|\dfrac{\partial \mathbf{r}}{\partial r}\right|} = \frac{\langle \cos\theta,\ \sin\theta,\ 0 \rangle}{\sqrt{\cos^2\theta + \sin^2\theta}} = \langle \cos\theta,\ \sin\theta,\ 0 \rangle$$

$$\mathbf{e}_\theta = \frac{\dfrac{\partial \mathbf{r}}{\partial \theta}}{\left|\dfrac{\partial \mathbf{r}}{\partial \theta}\right|} = \frac{\langle -r\sin\theta,\ r\cos\theta,\ 0 \rangle}{r\sqrt{\sin^2\theta + \cos^2\theta}} = \langle -\sin\theta,\ \cos\theta,\ 0 \rangle$$

$$\mathbf{e}_z = \frac{\dfrac{\partial \mathbf{r}}{\partial z}}{\left|\dfrac{\partial \mathbf{r}}{\partial z}\right|} = \frac{\langle 0,\ 0,\ 1 \rangle}{\sqrt{1}} = \langle 0,\ 0,\ 1 \rangle$$

where

$$h_r = 1$$
$$h_\theta = r$$
$$h_z = 1$$

Line Element:

$$ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_z^2 dz^2 = dr^2 + rd\theta^2 + dz^2$$

Volume Element:

$$dV = h_r h_\theta h_z dr^2 d\theta^2 dz^2 = rdrd\theta dz$$

Gradient of $\mathbf{A}(r, \theta, z)$:

$$\nabla \mathbf{A} = \frac{1}{h_r}\frac{\partial \mathbf{A}}{\partial r}\mathbf{e}_r + \frac{1}{h_\theta}\frac{\partial \mathbf{A}}{\partial \theta}\mathbf{e}_\theta + \frac{1}{h_z}\frac{\partial \mathbf{A}}{\partial z}\mathbf{e}_z$$

$$= \left\langle \frac{\partial \mathbf{A}}{\partial r}, \frac{1}{r}\frac{\partial \mathbf{A}}{\partial \theta}, \frac{\partial \mathbf{A}}{\partial z} \right\rangle$$

Divergence of $\mathbf{A}(r, \theta, z)$:

$$\text{div } \mathbf{A} = \frac{1}{h_r h_\theta h_z}\left(\frac{\partial}{\partial r}(A_1 h_\theta h_z) + \frac{\partial}{\partial \theta}(A_2 h_r h_z) + \frac{\partial}{\partial z}(A_3 h_r h_\theta) \right)$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r}(A_1 r) + \frac{\partial}{\partial \theta}(A_2) \right) + \frac{\partial}{\partial z}(A_3)$$

$$= \frac{A_1}{r} + \frac{\partial}{\partial r}(A_1) + \frac{1}{r}\frac{\partial}{\partial \theta}(A_2) + \frac{\partial}{\partial z}(A_3)$$

Curl of **A**(*r*, θ, *z*):

$$\text{curl } \mathbf{A} = \frac{\mathbf{e}_r}{h_\theta h_z}\left(\frac{\partial}{\partial\theta}(A_3 h_z) - \frac{\partial}{\partial z}(A_2 h_\theta)\right) + \frac{\mathbf{e}_\theta}{h_z h_r}\left(\frac{\partial}{\partial z}(A_1 h_r) - \frac{\partial}{\partial r}(A_3 h_z)\right)$$
$$+ \frac{\mathbf{e}_z}{h_r h_\theta}\left(\frac{\partial}{\partial r}(A_2 h_\theta) - \frac{\partial}{\partial\theta}(A_1 h_r)\right)$$

so

$$\text{curl } \mathbf{A} = \frac{\mathbf{e}_r}{r}\left(\frac{\partial}{\partial\theta}(A_3) - \frac{\partial}{\partial z}(A_2 r)\right) + \mathbf{e}_\theta\left(\frac{\partial}{\partial z}(A_1) - \frac{\partial}{\partial r}(A_3)\right)$$
$$+ \frac{\mathbf{e}_z}{r}\left(\frac{\partial}{\partial r}(A_2 r) - \frac{\partial}{\partial\theta}(A_1)\right)$$

and further

$$\text{curl } \mathbf{A} = \left(\frac{1}{r}\frac{\partial A_3}{\partial\theta} - \frac{\partial A_2}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial r}\right)\mathbf{e}_\theta + \left(\frac{A_2}{r} + \frac{\partial A_2}{\partial r} - \frac{\partial A_1}{\partial\theta}\right)\mathbf{e}_z$$

Laplacian of **A**(*r*, θ, *z*):

$$\nabla^2\mathbf{A} = \frac{1}{h_r h_\theta h_z}\left(\frac{\partial}{\partial r}\left(\frac{h_\theta h_z}{h_r}\frac{\partial\mathbf{A}}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{h_r h_z}{h_\theta}\frac{\partial\mathbf{A}}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(\frac{h_r h_\theta}{h_z}\frac{\partial\mathbf{A}}{\partial z}\right)\right)$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\frac{\partial\mathbf{A}}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\mathbf{A}}{\partial\theta}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial\mathbf{A}}{\partial z}\right)\right)$$

$$= \frac{1}{r}\frac{\partial\mathbf{A}}{\partial r} + \frac{\partial^2\mathbf{A}}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\mathbf{A}}{\partial\theta^2} + \frac{\partial^2\mathbf{A}}{\partial z^2}$$

APPENDIX B: SPHERICAL COORDINATES

Transformation:

$$x = r*\sin\theta\cos\phi$$
$$y = r*\sin\theta\sin\phi$$
$$z = r*\cos\theta$$

and

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$
$$\phi = \sin^{-1}\left(\frac{\sqrt{x^2 + y^2}}{r}\right) = \cos^{-1}\left(\frac{z}{r}\right)$$

Let $\mathbf{r} = (x, y, z)$.

Base Vectors and Scaling Factors:

$$\mathbf{e}_r = \frac{\langle \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta \rangle}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$$= \langle \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta \rangle$$

$$\mathbf{e}_\theta = \frac{\langle r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta \rangle}{r\sqrt{\cos^2\theta\cos^2\phi + r\cos^2\theta\sin^2\phi + r^2\sin^2\theta}}$$

$$= \langle r\cos\theta\cos\phi, r\cos\theta\sin\phi, -r\sin\theta \rangle$$

$$\mathbf{e}_\phi = \frac{\langle -r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0 \rangle}{r\sqrt{\sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi}}$$

$$= \langle -r\sin\theta\sin\phi, r\sin\theta\cos\phi, 0 \rangle$$

where

$$h_r = 1$$
$$h_\theta = r$$
$$h_\phi = r\sin\theta$$

Line Element:

$$ds^2 = h_r^2 dr^2 + h_\theta^2 d\theta^2 + h_\phi^2 d\phi^2 = dr^2 + rd\theta^2 + r^2\sin^2\theta d\phi^2$$

Volume Element:

$$dV = h_r h_\theta h_\phi dr d\theta d\phi = r^2\sin\theta dr d\theta d\phi$$

Gradient of $\mathbf{A}(r, \theta, \phi)$:

$$\nabla\mathbf{A} = \frac{1}{h_r}\frac{\partial\mathbf{A}}{\partial r}\mathbf{e}_r + \frac{1}{h_\theta}\frac{\partial\mathbf{A}}{\partial\theta}\mathbf{e}_\theta + \frac{1}{h_\phi}\frac{\partial\mathbf{A}}{\partial\phi}\mathbf{e}_\phi$$

$$= \left\langle \frac{\partial\mathbf{A}}{\partial r}, \frac{1}{r}\frac{\partial\mathbf{A}}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\mathbf{A}}{\partial\phi} \right\rangle$$

Divergence of $\mathbf{A}(r, \theta, \phi)$:

$$\text{div }\mathbf{A} = \frac{1}{h_r h_\theta h_\phi}\left(\frac{\partial}{\partial r}(A_1 h_\theta h_\phi) + \frac{\partial}{\partial\theta}(A_2 h_r h_\phi) + \frac{\partial}{\partial\phi}(A_3 h_r h_\theta) \right)$$

$$= \frac{1}{r^2\sin^2\theta}\left(\frac{\partial}{\partial r}(A_1 r^2\sin^2\theta) + \frac{\partial}{\partial\theta}(A_2 r\sin\theta) + \frac{\partial}{\partial\phi}(A_3 r) \right)$$

$$= \frac{2\sin\theta}{r}A_1 + \sin\theta\frac{\partial A_1}{\partial r} + \frac{\cos\theta}{r\sin^2\theta}A_2 + \frac{1}{r\sin\theta}\frac{\partial A_2}{\partial\theta} + \frac{1}{r\sin^2\theta}\frac{\partial A_3}{\partial\phi}$$

Curl of $\mathbf{A}(r, \theta, \phi)$:

$$\text{curl } \mathbf{A} = \frac{\mathbf{e}_r}{h_\theta h_\phi}\left(\frac{\partial}{\partial\theta}(A_3 h_\phi) - \frac{\partial}{\partial\phi}(A_2 h_\theta)\right)$$

$$+ \frac{\mathbf{e}_\theta}{h_\phi h_r}\left(\frac{\partial}{\partial\phi}(A_1 h_r) - \frac{\partial}{\partial r}(A_3 h_\phi)\right) + \frac{\mathbf{e}_\phi}{h_r h_\theta}\left(\frac{\partial}{\partial r}(A_2 h_\theta) - \frac{\partial}{\partial\theta}(A_1 h_r)\right)$$

so

$$\text{curl } \mathbf{A} = \frac{1}{r^2\sin^2\theta}\left(\frac{\partial}{\partial\theta}(A_3 r\sin\theta) - \frac{\partial}{\partial\phi}(A_2 r)\right)\mathbf{e}_r$$

$$+ \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}(A_1) - \frac{\partial}{\partial r}(A_3 r\sin\theta)\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(A_2 r) - \frac{\partial}{\partial\theta}(A_1)\right)\mathbf{e}_\phi$$

and further

$$\text{curl } \mathbf{A} = \frac{1}{r\sin^2\theta}\left(A_3\cos\theta + \sin\theta\frac{\partial A_3}{\partial\theta} - \frac{\partial A_2}{\partial\phi}\right)\mathbf{e}_r$$

$$+ \frac{1}{r\sin\theta}\left(\frac{\partial A_1}{\partial\phi} - A_3\sin\theta - r\sin\theta\frac{\partial A_3}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(A_2 + r\frac{\partial A_2}{\partial r} - \frac{\partial A_1}{\partial\theta}\right)\mathbf{e}_\phi$$

Laplacian of $\mathbf{A}(r, \theta, \phi)$:

$$\nabla^2\mathbf{A} = \frac{1}{h_r h_\theta h_\phi}\left(\frac{\partial}{\partial r}\left(\frac{h_\theta h_\phi}{h_r}\frac{\partial\mathbf{A}}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{h_r h_\phi}{h_\theta}\frac{\partial\mathbf{A}}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{h_r h_\theta}{h_\phi}\frac{\partial\mathbf{A}}{\partial\phi}\right)\right)$$

$$= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\mathbf{A}}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\mathbf{A}}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\mathbf{A}}{\partial\phi}\right)\right)$$

$$= \frac{1}{r^2}\left(r^2\frac{\partial^2\mathbf{A}}{\partial r^2} + 2r\frac{\partial\mathbf{A}}{\partial r} + \frac{\partial^2\mathbf{A}}{\partial\theta^2} + \cot\theta\frac{\partial\mathbf{A}}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2\mathbf{A}}{\partial\phi^2}\right)$$