# Wallpaper Groups

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## Introduction

A wallpaper group, or a two-dimensional crystallographic group, is a mathematical device used to describe and classify repetitive designs on two-dimensional surfaces ("Wallpaper Group"). The designs are classified into groups by looking at the types of symmetries present. Those which contain the same symmetries are put in the same group. Classifying is a process often used by mathematicians to organize data into more easily accessible and readily available information. Classifying patterns into groups allows us to identify and compare different patterns. For example, without such classifications, it might be thought that the two patterns shown below (Figure 1) were created in completely different manners when, in reality, they are classified as the same wallpaper group.



**Figure 1**

## Background

In order to understand what a wallpaper group is, it is important first to know what a group is. The definition of a group consists of four axioms. The first of these says that the set of elements in a group must be closed under the operation. In other words, when any two elements, including an element and itself, are combined, the result must also be an element in that group. For example, $\{1, 2, 3, 4\}$ under multiplication modulo 5 is a group whereas $\{1, 2, 3\}$ under multiplication modulo 4 is not a group. The

latter is not a group because two multiplied by two is four but four modulo four is zero, which is not in the set of elements; in other words, the set of elements is not closed.

The second axiom is the identity axiom. It says there has to be some element in the group called the identity, such that if the identity is applied under the operation to any element $x$ in the group, the result is $x$. For the set of positive rational numbers under multiplication, the identity element is one. One multiplied by any rational number simply returns that number. As another example, for the integers under addition, the identity element is zero.

The third axiom is the inverse axiom. Its requirement is that each element in the group must have an inverse also in the group such that when the two are combined under the operation, the result is the identity. Looking again at multiplication of the positive rational numbers, the inverse of any of these numbers would be its reciprocal. For example, the inverse of two is one half. When these two numbers are multiplied, the result is the identity for multiplication, or one.

The associative axiom is the fourth axiom in the definition of a group. This axiom states that when three elements are combined together under the operation, the computation can be carried out in any manner, as long as the order of the elements remains the same, and the same result is obtained. For example, given {1, 2, 3, 4} under multiplication modulo 5, (1 x 2) x 3 is equal to 1 x (2 x 3). The placement of the parentheses is different yet both sides of the equation equal 1; we are assured of this because multiplication modulo 5 on {1, 2, 3, 4} is associative.

There are many different types of groups. In the context of this paper, symmetries of a pattern form what mathematicians call a symmetry group (Joyce). Wallpaper groups

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are a type of symmetry group. The elements in a wallpaper group are called transformations. The transformations used to create wallpaper groups each preserve distance and are thus termed planar isometries. The four types of transformations used are translations, reflections, rotations, and glide reflections.

- A translation slides the pattern a certain distance in a specific direction.
- Reflections fix one line in the plane, called the axis of reflection, and exchange points on one side of the axis with points on the other side of the axis at the same distance from the axis and such that the line through the points is perpendicular to the axis of reflection (Joyce). A reflection can be thought of as a flip. To reflect something over a line is like flipping it over that line.
- Rotations fix one point in the plane and then turn the plane some angle around that point (Joyce). The order of rotation is how many times this must be done before the pattern is back to where it started.
- A glide reflection is a composition of a reflection across an axis and a translation along the axis (Joyce). An example of this is footprints in the sand (see Figure 2).
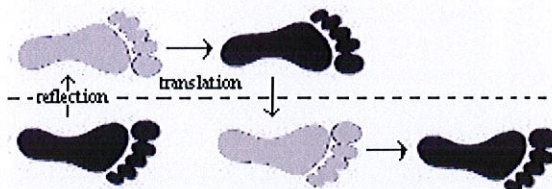


**Figure 2: Glide Reflection**

The operation used in symmetry groups is composition of the transformations. When two symmetries are composed, the result is a third symmetry also contained in the group.

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In other words, when any two of the four transformation types are composed, the resulting transformation is one of the original four transformation types. This means that wallpaper groups pass the first group axiom. In other words, the operation has closure.

The second axiom that wallpaper groups need to pass in order to truly be groups is the identity axiom. The identity of wallpaper groups is simply the identity transformation, a transformation that does nothing to the plane, thus leaving the pattern the same.

The inverse axiom is a bit more complicated. The inverse of a wallpaper group undoes, or does the opposite of, what was first done. If an object was rotated 120 degrees in one direction, the inverse would be to rotate it 120 degrees in the opposite direction. The inverse of a reflection, on the other hand, is itself that same reflection.

Wallpaper groups also pass the last requirement of associativity, simply because composition of functions (transformations) is itself associative. Having passed all four axioms, wallpaper groups are appropriately defined as groups.

Wallpaper groups are known by many different names: plane symmetry groups, planar wallpaper symmetry groups, two-dimensional crystallographic groups, two-dimensional space groups, and wallpaper patterns. One person even went as far as to shorten the name to wallpattern.


**How Many Distinct Wallpaper Groups?**

There are exactly seventeen distinct wallpaper groups generated by the four isometric transformations. A question often asked is "why are there only seventeen wallpaper groups?" In the article "The 17 Plane Symmetry Groups", R. Schwarzenberger

gives a proof which answers this question. In order to do this, he introduces four invariants: the lattice, the point group, the action of the point group on the lattice, and the shift vectors (Schwarzenberger 124). A lattice is a set of translation vectors (see Figure 3). There are five different types of lattices: parallelogram, rectangular, rhombic, square, and hexagonal. Each wallpaper group has a specific lattice.
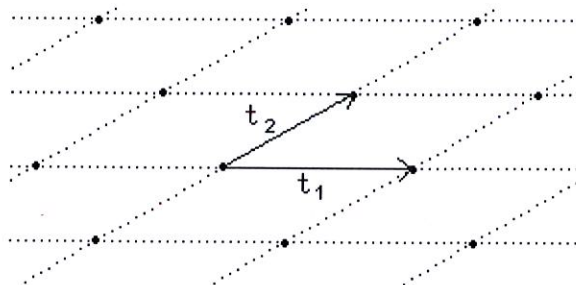


**Figure 3: Parallelogram Lattice with Translation Vectors $t_1$ and $t_2$**

A point group is a set of operations which leaves at least one point unmoved (Weisstein). This includes rotations and reflections. The action of the point group on the lattice is the application of the operations to the lattice such that at least one point is not moved. In Figure 4, Rectangle ABCD is rotated 180 degrees counterclockwise. In this case, the operation is rotation and the point unmoved is point D.
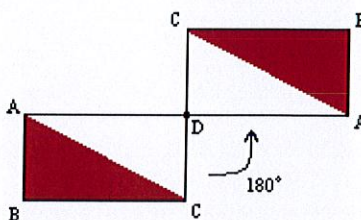


**Figure 4**

Shift vectors measure the extent to which a reflection or rotation must be combined with a translation not in the lattice to obtain symmetry of the symmetry group (Schwarzenberger 125). For example, glide reflections are a combination of reflections and translations.

Schwarzenberger defines a wallpaper group using two requirements. The first is that there must exist two linearly independent vectors such that the lattice equals the set of all linear combinations of the two (125). The second is that the point group is finite (125).

He goes on to explain how the "crystallographic restriction" requires that symmetry must be present in a lattice; therefore only one, two, three, four, and six-fold symmetry axes are possible. Eric Weisstein shows why there are no other possibilities by looking at the sum of the interior angles of a regular polygon divided by the number of sides. Since symmetry must be present, this quantity must be a divisor of 360 degrees. In other words,

$$\frac{180°\,(n-2)}{n} = \frac{360°}{m},$$

where n is the number of sides of the polygon, m is a divisor of 360 degrees, and m and n are both integers. By simplifying, we obtain the equation

$$\frac{2n}{n-2} = m,$$

(Weisstein)

Since, by definition, a polygon must have three or more sides, we will only look at the cases when $n \geq 3$. The formula works when $n = 3$ and when $n = 4$; however, when $n = 5$, m is a non-integer. This rules out the possibility of five-fold symmetry. When $n = 6$, $m = 3$ which satisfies the requirement. Any other regular polygons for which the formula would work would have to have seven or more sides, and yield values of m less than three. (Note that $2n/(n-2)$ is a decreasing sequence for $n=3,4,5,...$ with a limit of 2). So the only two integers left that m could possibly equal are 1 and 2; and these two cases for m both lead to impossible values for n (with n = number of sides). So the only

possibilities for symmetry are one-fold, two-fold, three-fold, four-fold, and six-fold symmetry.

For a more visual example, a pattern may be made using only regular triangles because the measure of the angles add up to 360 degrees (see Figure 5a.). This would be an example of six-fold symmetry. A pattern made only from regular pentagons (five-fold symmetry), however, has an interior angle measure of 108 degrees, which does not divide 360 degrees (see Figure 5b.).
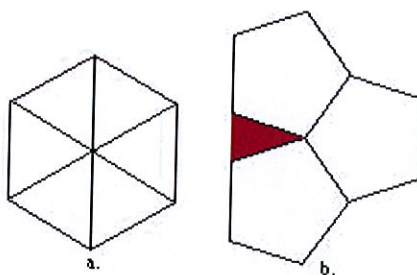


**Figure 5**

Because of the crystallographic restriction and the two requirements of a wallpaper group, the invariants which can occur are extremely limited (Schwarzenberger 125).

In his paper, Schwarzenberger proves that the seventeen wallpaper groups can be divided into three classes: those with no reflections, those with one reflection, and those with more than one reflection (see Figure 6).

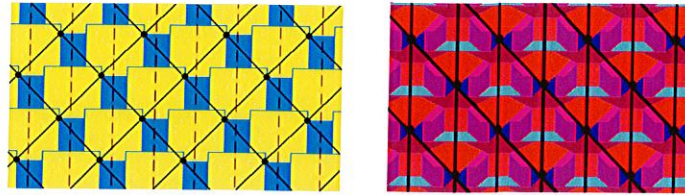|  | Wallpaper Groups |
|---|---|
| No Reflections | p1, p2, p3, p4, p6 |
| One Reflection | cm, pm, pg |
| More than One Reflection | cmm, pmm, pmg, pgg, p31m, p3m1, p4mm, p4mg, p6mm |

**Figure 6**

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| Internat'l (short) | Pólya; Guggenheimer | Niggli | Speiser | Fejes Tóth; Cadwell | Shubnikov-Koptsik | Wells Bell & Fletcher |
|---|---|---|---|---|---|---|
| $p1$ | $C_1$ | $C_1^1$ | $C_1$, Abb. 17 | $W_1$ | $(b/a)1$ | 1 |
| $p2$ | $C_2$ | $C_2^{I}$ | $C_2$, Abb. 18 | $W_2$ | $(b/a){:}2$ | 2 |
| $pm$ | $D_1kk$ | $C_s^{I}$ | $C_s^{I}$, Abb. 19 | $W_1^2$ | $(b{:}a){:}m$ | 3 |
| $pg$ | $D_1gg$ | $C_s^{II}$ | $C_s^{II}$, Abb. 20 | $W_1^3$ | $(b{:}a){:}\bar{b}$ | 4 |
| $cm$ | $D_1kg$ | $C_s^{III}$ | $C_s^{III}$, Abb. 21 | $W_1^1$ | $(a/a)/m$ | 8 |
| $pmm$ | $D_2kkkk$ | $C_{2v}^{I}$ | $C_{2v}^{I}$, Abb. 22 | $W_2^2$ | $(b{:}a){:}2{\cdot}m$ | 5 |
| $pmg$ | $D_2kkgg$ | $C_{2v}^{III}$ | $C_{2v}^{III}$, Abb. 24 | $W_2^3$ | $(b{:}a){:}m{:}\tilde{a}$ | 6 |
| $pgg$ | $D_2gggg$ | $C_{2v}^{II}$ | $C_{2v}^{II}$, Abb. 23 | $W_2^4$ | $(b{:}a){:}\bar{b}{:}\tilde{a}$ | 7 |
| $cmm$ | $D_2kgkg$ | $C_{2v}^{IV}$ | $C_{2v}^{IV}$, Abb. 25 | $W_2^1$ | $(a/a){:}2{\cdot}m$ | 9 |
| $p4$ | $C_4$ | $C_4^{I}$ | $C_4$, Abb. 26 | $W_4$ | $(a{:}a){:}4$ | 10 |
| $p4m$ | $D_4$ | $C_{4v}^{I}$ | $C_{4v}^{I}$, Abb. 27 | $W_4^1$ | $(a{:}a){:}4{\cdot}m$ | 11 |
| $p4g$ | $D_4$ | $C_{4v}^{II}$ | $C_{4v}^{II}$, Abb. 28 | $W_4^2$ | $(a{:}a){:}4{\odot}\tilde{a}$ | 12 |
| $p3$ | $C_3$ | $C_3^{I}$ | $C_3$, Abb. 29 | $W_3$ | $(a/a){:}3$ | 13 |
| $p3m1$ | $D_3$ | $C_{3v}^{I}$ | $C_{3v}^{II}$, Abb. 31 | $W_3^1$ | $(a/a){:}m{\cdot}3$ | 15 |
| $p31m$ | $D_3$ | $C_{3v}^{II}$ | $C_{3v}^{I}$, Abb. 30 | $W_3^2$ | $(a/a){\cdot}m{\cdot}3$ | 14 |
| $p6$ | $C_6$ | $C_6^{I}$ | $C_6$, Abb. 32 | $W_6$ | $(a/a){:}6$ | 16 |
| $p6m$ | $D_6$ | $C_{6v}^{I}$ | $C_{6v}$, Abb. 33 | $W_6^1$ | $(a/a){:}m{\cdot}6$ (some alternatives exist) | 17 |

**Figure 7: Chart of Notation Types of Wallpaper Groups**

## Notation

The seventeen wallpaper groups are notated in many different ways (see Figure 7). The international notation, seen in Figure 6, is used most often. Each letter and number in this notation has a unique meaning. The symbols p and c stand for the type of cell the pattern has. A primitive cell, p, is a region which generates the pattern using only translations (Edwards). Its lattice, or the set of translation vectors, can be in the shape of a parallelogram, rectangle, square, rhombus, or hexagon. Primitive cells account for fifteen of the seventeen wallpaper groups. A centered cell, c, has a rhombic lattice and twice the area of a primitive cell (see Figure 8) (Edwards). It is used only in groups cm and cmm (Edwards).

**Figure 8: Notice the rhombic lattice in a. (centered cell)
and the parallelogram lattice in b. (primitive cell).**

The number following the p or c is the highest order of rotation of the wallpaper group (Edwards). For example, group p3 has an order of rotation of three (see Figure 9). In other words, the fundamental region can be rotated 120 degrees, one third of 360 degrees, and still look the same.
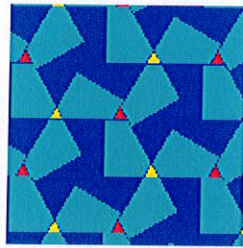


**Figure 9: Example of Wallpaper Group p3**

If the wallpaper group has a mirror reflection perpendicular to an edge of the cell it is notated with an m (Edwards). A glide reflection is notated with a g. If there is no symmetry axis perpendicular to an edge of the cell, a 1 is used for the notation (Edwards). The last symbol is for a symmetry axis at an angle to an edge of the cell (Edwards).

## Classifications

Provided below are two tools used to classify patterns as wallpaper groups. Figure 10 is a chart listing the properties of each group. Figure 11 is a flow chart which asks questions about the operations in order to determine the type of wallpaper group.

| TYPE | REFLECTION | ROTATION | ORDER OF ROTATION | GLIDE REFLECTION | SPECIAL CHARACTERISTICS | LATTICE |
|---|---|---|---|---|---|---|
| p1 | No | No | NA | No | Only translations | Parallelogram |
| p2 | No | Yes | 2 | No | 4 types of rotations | Parallelogram |
| Pm | Yes | No | NA | No | Reflection axes are parallel | Rectangle |
| Pg | No | No | NA | Yes | | Rectangle |
| Cm | Yes | No | NA | Yes | Reflection axes are parallel | Rhombus |
| Pmm | Yes | Yes | 2 | No | Reflection axes are perpendicular | Rectangle |
| Pmg | Yes | Yes | 2 | Yes | Reflection axes are parallel | Rectangle |
| Pgg | No | Yes | 2 | Yes | | Rectangle |
| Cmm | Yes | Yes | 2 | Yes | Reflection axes are perpendicular | Rhombus |
| p4 | No | Yes | 4 | No | | Square |
| p4m | Yes | Yes | 4 | Yes | Reflection axes intersect at 45 degrees | Square |
| p4g | Yes | Yes | 4 | Yes | Reflection axes are perpendicular | Square |
| p3 | No | Yes | 3 | No | | Hexagon |
| p31m | Yes | Yes | 3 | Yes | Reflection axes intersect at 60 degrees Centers of rotation are on reflection axes 2 types of centers of rotation | Hexagon |
| p3m1 | Yes | Yes | 3 | Yes | Reflection axes intersect at 60 degrees 3 types of centers of rotation | Hexagon |
| p6 | No | Yes | 6 | No | | Hexagon |
| p6m | Yes | Yes | 6 | Yes | Reflection axes intersect at 30 degrees | Hexagon |

**Figure 10: The Seventeen Wallpaper Groups and their Properties (Edwards)**
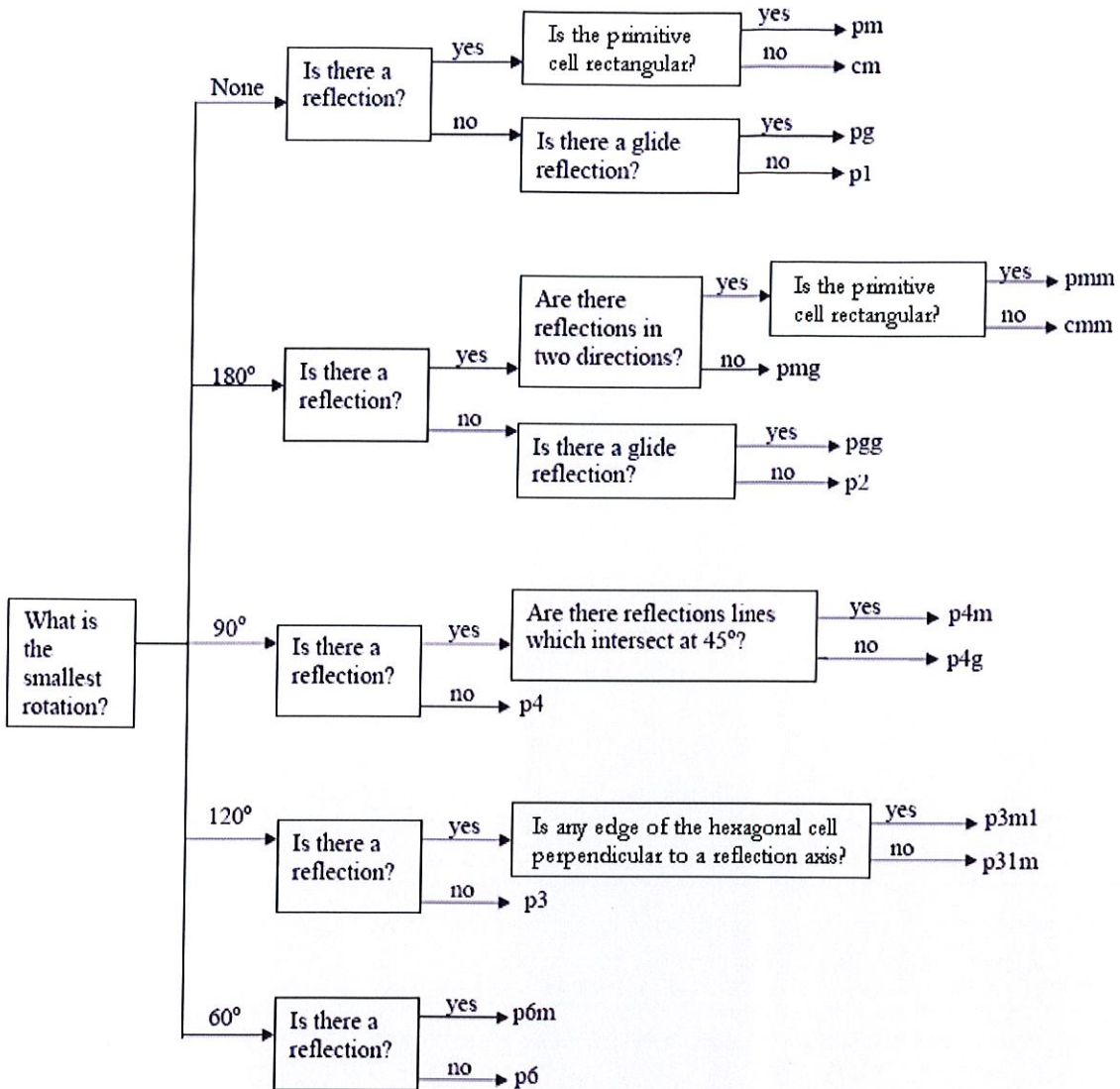
**Figure 11: Wallpaper Group Flow Chart (Aslaksen)**

**History**

Wallpaper groups can be seen in art, nature, and architecture around the world. Many cultures have used them to decorate their homes and buildings. For example, the Alhambra, a walled city and fortress in Granada, Spain, is lavishly decorated with wallpaper groups (Addington). It was built during the Nasrid Dynasty (1238-1492) (Addington). At this time, Spain was an intellectual center for science and mathematics (Addington). Because Islamic art did not allow for the use of representations of living beings, geometric patterns, especially symmetric patterns, were used to decorate the floors, ceilings, and walls of the Alhambra (see Figure 12) (Kieft). It is thought that all seventeen possible wallpaper groups can be found there.
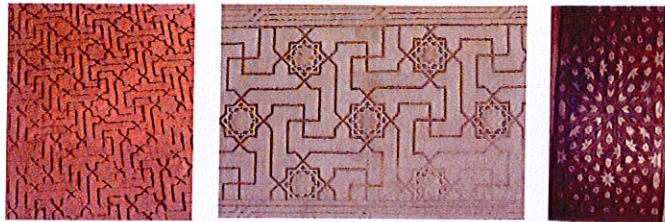


**Figure 12: Pictures taken at the Alhambra**

Because of this beautiful display in the palace at Granada, M.C. Escher, a Dutch graphic artist who, incidentally, was never formally trained in mathematics, became fascinated in the art of symmetric patterns (Kieft). Escher experimented with plane-filling techniques, shapes, and transformations (Kieft). He is very well known for his use of tessellations in his prints (see Figure 13).

A tessellation is a repeating pattern of interlocking shapes. Although quite similar to wallpaper groups, tessellations are not restricted to using only specific polygons for the lattice as wallpaper groups are. Mathematicians tend to be very interested in tessellations because of their ties to symmetry of figures, angle divisions, rotation of objects, and other

various geometrical concepts (Kieft).  Some of Escher's prints have been used to study

visual perception in fields like physics, geology, chemistry, and psychology (Kieft).
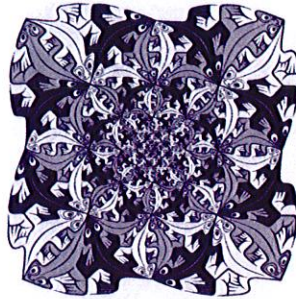


**Figure 13: "Smaller and Smaller" by Escher**

Camille Jordan (1838-1922), Evgraf Stepanovich Federov (1853-1919), Artur

Moritz Schoenflies (1853–1928), and William Barlow (1845–1934) were the main

contributors to the classifications of both two-dimensional (wallpaper groups) and three-

dimensional crystallographic groups.  The word "group" was first used in 1869 by

Jordan.  He was interested in the ways direct symmetry operations could be combined.

(Lalena)

In the late 1860s, French mathematician Camille Jordan discovered sixteen of the

seventeen wallpaper groups (Joyce). Later that century, Evgraf Stepanovich Fedorov,

Arthur Moritz Schoenflies, and William Barlow classified all seventeen wallpaper groups

(Joyce).  Even though they were classified in the 1800s, most of the wallpaper groups

could be found at the Alhambra, which was built long before then.

The 230 three-dimensional crystallographic groups were also derived by Fedorov,

Schoenflies, and Barlow in the late 1800s.  Although all three men reached the same

conclusion, they all managed to do so using different strategies.  Fedorov looked at the

physical properties, Schoenflies studied the geometry, and Barlow focused on structure.

As a post-graduate student in Germany, Arthur Schoenflies established that the groups could be derived. He published his work as he progressed through the derivation. Fedorov, a Russian mathematician and crystallographer, came across the publications and decided to also attempt to derive the three-dimensional crystallographic groups. He sent his results to Schoenflies and indicated some inaccuracies in Schoenflies' derivation. In turn, Schoenflies did the same. From this moment on, they entered into a lively correspondence. (Galiulin 904)

In 1890, Fedorov was the first to derive the 230 three-dimensional crystallographic groups (904). However, Schoenflies was not far behind and also derived them soon after. In the book Fifty Years of X-ray Diffraction, the strategies used by Fedorov and Schoenflies to derive the 230 three-dimensional crystallographic groups are compared.

> […] for Schoenflies it is just an interesting case of representation in the theory of groups, in particular infinite groups, which were being developed at that time; for Fedorov it is a means of studying real systems of configurations, the underlying feature of a crystal. Fedorov found his results by deriving the only possible 230 types of basic design which underlie all natural crystals.

(Andrade 344)

Schoenflies obtained the 230 arrangements but, coming from the morphological side of crystallography, he attributed physical significance to the polyhedral fundamental domains (called by him stereohedra) and thus distinguished between the actually possible ('real') and the other 'asymmetric' space groups.

(Andrade 352)

William Barlow, on the other hand, had a remarkable ability to spatially visualize atomic structures in three dimensions. Using this ability, he initially looked at the

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possible structures of well-known compounds and was eventually able to deduce that for all compounds there are just 230 different kinds of symmetry arrangements which form the basis for crystal structures. Though he was beaten to publication by Federov and Schoenflies, Barlow's explanation, when it was published, showed yet another strategy of arriving at the same conclusion. (Tandy)

**Application to Teaching**

Wallpaper groups are hardly ever mentioned in high schools even though they would provide students with much practice in identifying and classifying symmetry. They could easily be incorporated into a Geometry curriculum. Tessellations are often discussed in Geometry, and M.C. Escher's works are frequently displayed on the walls of art and mathematics classrooms. Since wallpaper groups are a type of tessellation, they could easily be tied into that section. Also, the crystallographic restriction uses the concept of the measure of the interior angles of a regular polygon, which is typically discussed in Geometry. Wallpaper groups could even be presented in the lower grades. It would introduce children to the concept of patterns.

Below is an example unit that may be used to teach this concept. Worksheets, handouts, lesson plans, and overheads are provided in the appendix.

**Wallpaper Groups: Geometry and Symmetry**

This unit should be used as a review or expansion of symmetry in a geometry class. It should take approximately five days to complete.

Standards

The National Council for Teachers of Mathematics Standards covered in this Unit can be found in the appendix (A1).

Student Prerequisites

Students should understand the terms: translation, reflection, line of symmetry, rotation, order of rotation and glide reflection, and know how to identify each.

- A **translation** slides a figure a certain distance in a specific direction.
- A **reflection** can be thought of as a flip. To reflect something over a line is like flipping it over that line. If a line can be drawn through a figure so that one side is the exact same as the other side, that figure has reflectional symmetry. The line that divides it is called the **line of symmetry**.
- **Rotations** turn a figure around a fixed point. The order of rotation is how many times this can be done before the figure is back to where it started.
- A **glide reflection** is a composition of a reflection across an axis and a translation along the axis (Joyce).

Unit Objectives

Students will be able to:
- identify the changes each symmetry operation induces.
- determine the types of symmetry present in a pattern.
- identify the seventeen different types of wallpaper groups.
- apply the ideas of symmetry and wallpaper groups to create their own design.
- clearly communicate their thoughts and ideas to their group members and classmates.

Unit Outline

Day 1: Review of Symmetries and Introduction of Wallpaper Groups

Day 2: A Closer Look at Wallpaper Groups

Day 3: Drawing Using Symmetry

Day 4: Design Your Own!

Day 5: Paper Presentations

*Extension: Designing Wallpaper Groups Using Kali

- This lesson requires some work be done well in advance.

**Lesson Plan for Day 1**

- Objectives
  - Students will:
    - be introduced to the seventeen different types of wallpaper patterns.
    - review and expand the concepts of reflection, rotation, glide reflection, and translation.
    - determine the types of symmetries in a pattern.
    - classify patterns as wallpaper groups by using their properties.

- Materials
  - Overheads and handouts provided in appendix
  - Overhead projector
  - Pen/pencil

- Procedure

  1. Review the types of symmetry.

     a. Using the overhead provided in the appendix (A-2), ask students to identify the types of symmetry present in each pattern. If there is a rotation, make sure they tell you the order of rotation. The answer key is also provided in the appendix (A-3).

     b. The patterns provided are wallpaper groups, two of which are from the same group. Tell the students that two of the four patterns are alike. Ask them to figure out which two are alike and why they are.

  2. Introduction of Wallpaper Groups

     a. Ask students if they were given a whole bunch of patterns like those they just looked at, how could they tell which ones were alike. They should respond that they would need to figure out what types of symmetry are present first; the patterns with all of the same symmetries are the ones that are alike.

     b. Ask them how many different types of patterns they think there are. Let them throw out a few ideas then tell them that there are actually only seventeen!!

- Evaluation
  - Hand out worksheet 1, provided in the appendix (A-4), as homework. They must identify all symmetries present and state the order of rotation if a rotation is present. Answer key is also provided in the appendix (A-5). Allow them to work in groups.

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**Lesson Plan for Day 2**

- Objectives
  - Students will:
    - identify similarities and differences of the wallpaper groups.
    - organize all of the properties of the wallpaper groups into an easy to use chart.
    - relate wallpaper groups in math to wallpaper groups in art/architecture.

- Materials
  - Overheads and worksheets provided in the appendix.
  - Overhead projector
  - Pen/pencil

- Procedure

  1. A Closer Look at the Wallpaper Groups

     a. Students should have completed worksheet 1 from the previous day. Ask them if they found any wallpaper groups that had that same properties, or symmetries. They should notice that cmm and pmg, p4m and p4g, and p3m1 and p31m have the same symmetries.

     b. Use the overheads in the appendix (A-6) to compare each of the pairs of similar wallpaper groups.

        i. When comparing cmm and pmg, point out how the reflection axes, or lines of symmetry, in pmg are parallel to each other; whereas, the reflection axes in cmm are perpendicular to each other.

        ii. The number of reflection axes differs in p4m and p4g. There are four reflection axes in p4m and only two in p4g.

        iii. Using the overhead, show the students how the translations are parallel to the reflection axes in p31m but not in p3m1.

     c. Hand out the blank chart provided in the appendix (A-7). Put one on the overhead and have the students help fill it in. They will need to hold on to their chart to use on their assignment. A completed chart is also provided in the appendix (A-8).

- Evaluation
    - Handout worksheet 2, provided in the appendix (A-9), as homework. They may use their chart to figure out which wallpaper group each pattern belongs to. The answer key is provided in the appendix (A-10). Allow them to work in groups of two or three.

    - Have each student find examples of the wallpaper groups outside of the classroom (city streets, parks, buildings, etc.) and write a 1-2 page paper on which group they saw, how they knew it was that group, where they saw it, and how it added to the beauty of the object. This should not be due until Day 5. On Day 5, each student will be required to read his or her paper in front of the class.


**Lesson Plan for Day 3**

- Objectives
    - Students will apply symmetry operations to figures.

- Materials
    - Grid Paper provided in the appendix
    - Pencil/Pen
    - Protractors
    - Overheads and Worksheets provided in the appendix
    - Overhead Projector

- Procedure

    1. Drawing using Symmetry.

        a. Use the Drawing Symmetric Figures Overhead (A-11) to show students how each symmetry operation is performed. Have the students define each symmetry operation before you show them how to perform it.

        b. Show the students how the grid paper can be used as a guide when drawing the figures.

            i. The grid paper is very useful in maintaining the size of the figure for all symmetry operations. The new figure should contain the same number of squares as the original figure.

            ii. For a reflection, the points on the original figure will be the same distance, or the same number of squares, from the reflection line as the new figure.

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iii. After a translation, each point on the original figure will be the same distance, or same number of squares, from the corresponding point on the new figure.

iv. A glide reflection is simply a combination of a reflection and translation.

v. A protractor should be used when performing a rotation. Each side should be rotated the same number of degrees.

- Evaluation
  o Handout Worksheet 3 (A-12) for practice drawing figures using each of the symmetry operations. They should hand it in at the end of the hour. The Answer Key is provided in the appendix (A-13).

  o Remind students that their papers from Day 2 are due in two days!

**Lesson Plan for Day 4**

- Objectives
  o Students will apply the ideas of symmetry and wallpaper groups to create their own design.

- Materials
  o Pencil
  o Completed chart of wallpaper group properties
  o Markers, colored crayons, or colored pencils
  o Rulers
  o Protractors
  o Grid paper provided in the appendix
  o Construction paper
  o Glue sticks

- Procedure

  1. Set up

     a. Push desks into groups of four if no tables are available. Section off the materials onto each group of desks. Ample grid paper should be provided (A-15).

  2. Design Your Own!

a. Have students get into groups of four and get out their completed chart of wallpaper group properties. Hand back their worksheet from the previous day.

b. Distribute the Example of Creating Your Own Design handout provided in the appendix (A-14). Explain to the students how they can combine the symmetry operations to make a pattern.

- Evaluation
  - o Have students pick one of the seventeen wallpaper groups and create their own design. Their final design should be glued onto construction paper and colored in with markers, colored crayons, or colored pencils. If they do not finish in class, it is homework due the following day.

  - o Remind students that their papers from Day 2 are due the next day!

## Lesson Plan for Day 5

- Objective
  - o Students will clearly communicate and present their discoveries of wallpaper groups in art/architecture to their classmates.

- Materials
  - o None

- Procedure

  1. Paper Presentations

     a. Call on students alphabetically by last name or numerically by student ID numbers to present.

     b. Students should stand in front of the classroom and read their papers one by one.

- Evaluation
  - o Students should have found at least one example of a wallpaper group outside of class. They should have named the group it belongs to and explained how they determined which group that was. They should tell where they found it and discuss how it added to the beauty of the object on which it was found.

**\*Extension Lesson Plan**

- Objective
    - Students will obtain assisted, hands-on practice creating patterns for each of the wallpaper groups.

- Materials
    - Computer Lab
    - Projector for computer (optional)

- Procedure

    1. Do in Advance!

        a. Make sure the computer lab is reserved!! Check with the laboratory technician to see if it is alright to download the program Kali from the website http://www.geometrygames.org/ (Weeks). If it is, download the program onto as many computers as you have students.

        b. Have the computers set up and ready to go when class begins.

        c. Optional: Set up a projector so what is displayed on the computer screen is projected onto a screen on the wall.

    2. Designing Wallpaper Groups Using Kali

        a. Have students gather around your computer. Show them how to access Kali.

        b. Explain to them that the notation used in the program is different from that they have been learning. What they learned was just one of many different ways to notate wallpaper groups. Hand out the conversion chart provided in the appendix (A-16).

        c. The program, Kali, allows the students to pick a specific wallpaper group and draw a picture on the screen while it performs the operations determined by that wallpaper group. Show the students an example of this on your computer.

        d. Show the students how to copy their drawings simply by going to Edit then Copy. Show them this on your computer and copy your drawing to Microsoft Word, or Microsoft Works.

        e. Allow them to practice making different designs on their own computers. Walk around to make sure they are doing what they are supposed to be doing and are not on the Internet.

- Evaluation
  - Students must make a design for each of the seventeen wallpaper groups and copy them to Microsoft Word, or Microsoft Works. Each design should be labeled with the notation they learned in class. When finished, they should pick their favorite design and print it. Before they can hand it in, they must first show you that they completed all seventeen.

Suggested Follow-up/Additional Activities

- Have students get into groups of four and give each group an example of a wallpaper group. Have them identify which wallpaper group it belongs to. When they have successfully identified it, give them a new one. This will give them more practice identifying the wallpaper groups and their symmetries.

- More examples of wallpaper groups can be found at http://www2.spsu.edu/math/tile/symm/types/index.htm

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# APPENDIX

# Appendix Outline

Standards

The NCTM standards addressed in this lesson are:

- Geometry

    o Analyze characteristics and properties of two- and three-dimensional geometric shapes.
      - Analyze properties and determine attributes of two- and three-dimensional objects. (NCTM 308)

    o Apply transformations and use symmetry to analyze mathematical situations.
      - Understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches.
      - Use various representations to help understand the effects of simple transformations and their compositions. (NCTM 308)

    o Use visualization and geometric modeling to solve problems.
      - Use geometric ideas to solve problems in, and gain insights into, other disciplines and other areas of interest such as art and architecture. (NCTM 308)

- Connections

    o Recognize and apply mathematics in contexts outside of mathematics. (NCTM 354)

- Communication

    o Communicate their mathematical thinking coherently and clearly to peers, teachers, and others. (NCTM 348)

# Examples of Wallpaper Groups

**1.**

**2.**

**3.**

**4.**
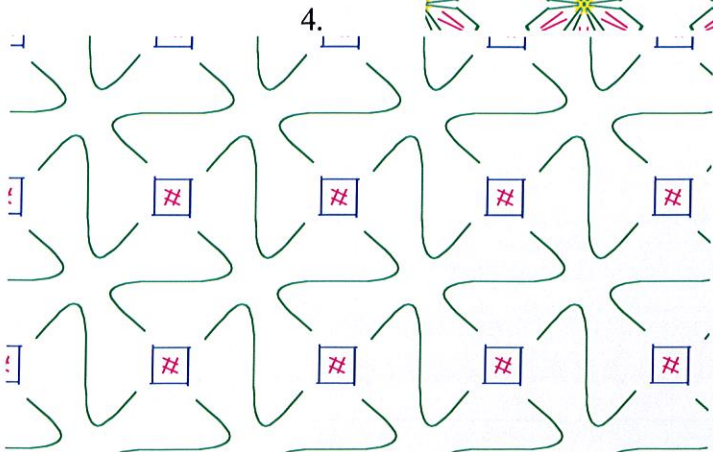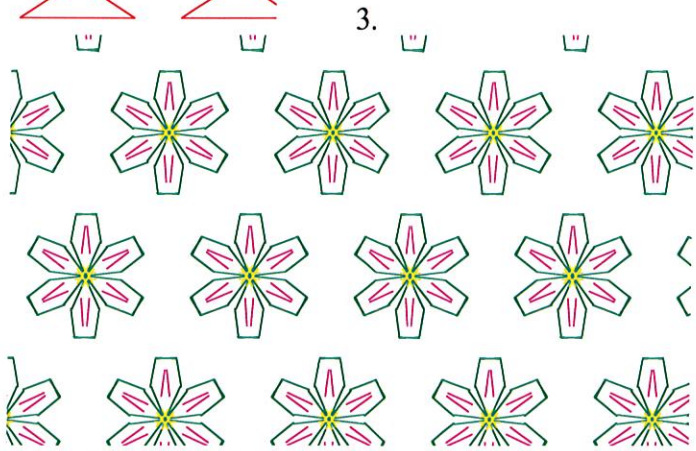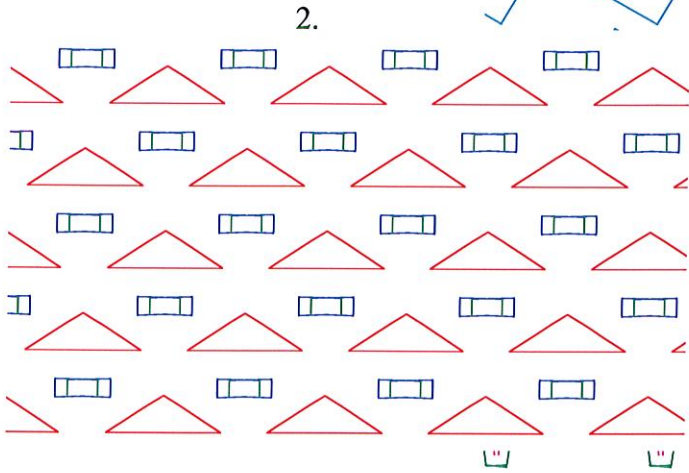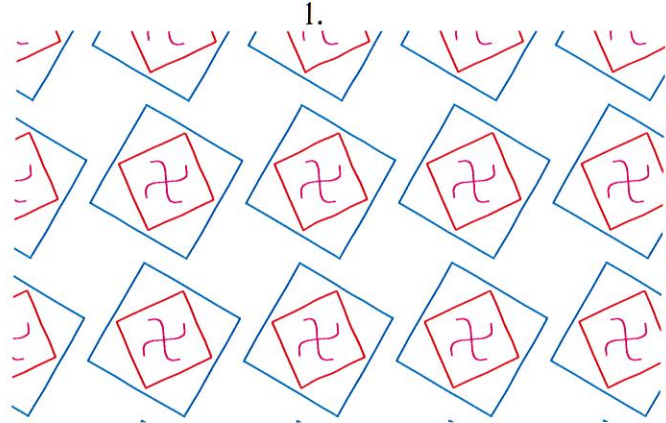
# Answer Key to Examples of Wallpaper Groups

1. **translation**
   **rotation – order 4**

2. translation
   reflection
   glide reflection

3. translation
   reflection
   rotation – order 6
   glide reflection

4. **translation**
   **rotation – order 4**

**Numbers 1 and 4 are the same wallpaper group because they have the same properties.**

## Worksheet 1: Identifying the Properties of Wallpaper Groups

For each pattern, list the types of symmetry present (rotation, reflection, glide reflection, translation). If a rotation is present, also state the order of rotation.

1.                p2

_____

_____

_____

_____

2.                pmm

_____

_____

_____

_____

3.                p3

_____

_____

_____

4.                p3m1

_____

_____

_____

5. <u>p4</u>

6. <u>p4m</u>

_____

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_____

7. <u>p6</u>

8. <u>p6m</u>

_____

_____

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_____

6
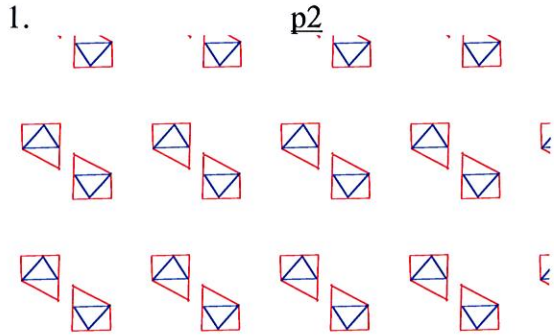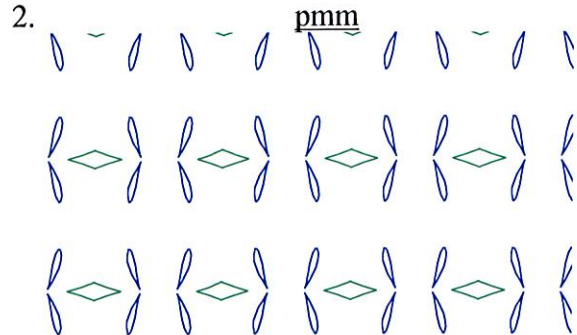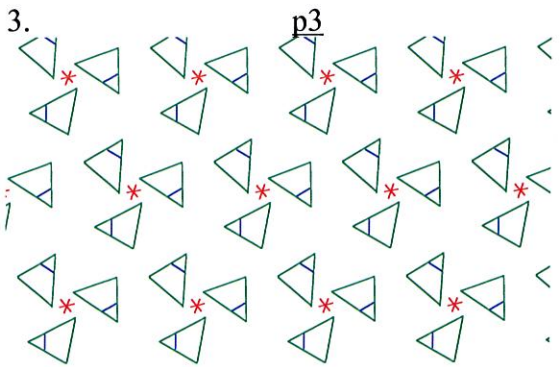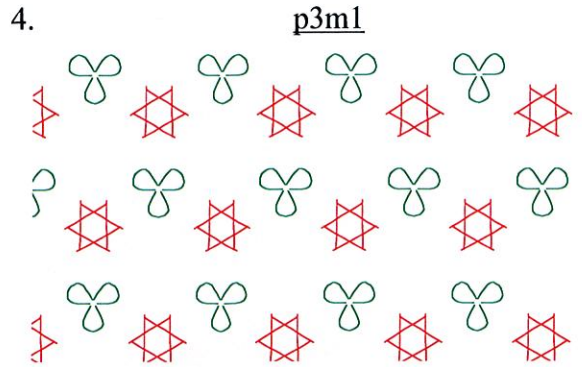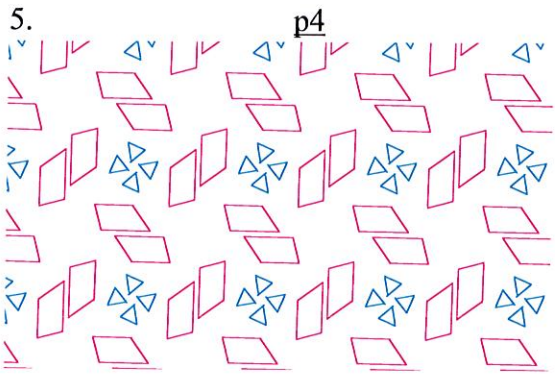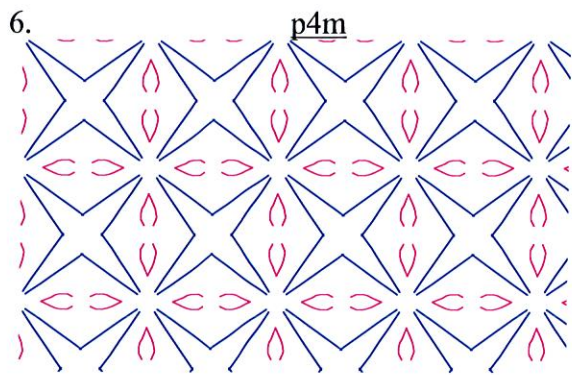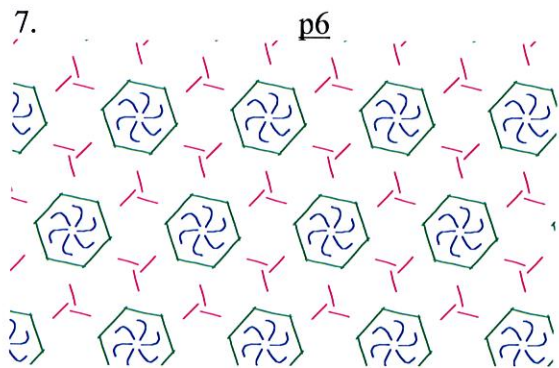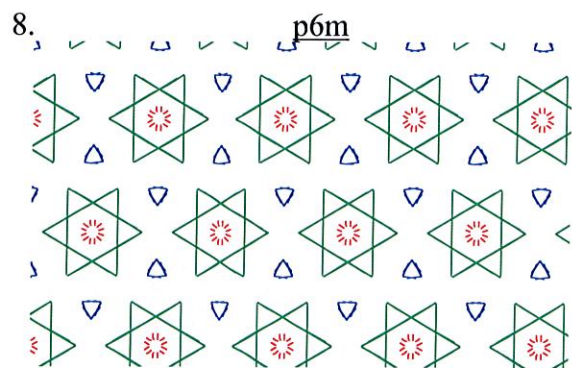
9.              p4g

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_____

_____
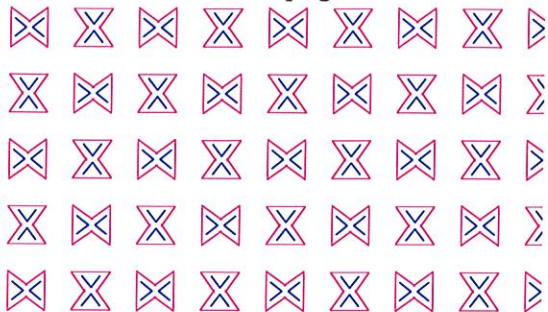
10.            p31m

_____

_____

_____

_____

11.            cmm

_____

_____

_____

_____

12.            pmg

_____

_____

_____

_____

7
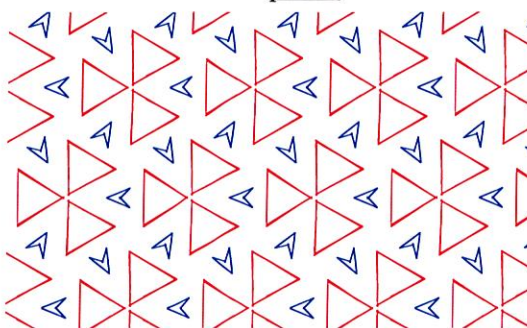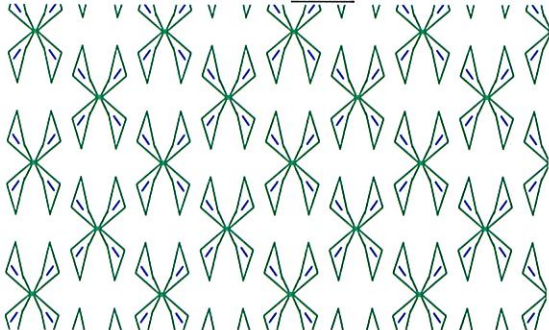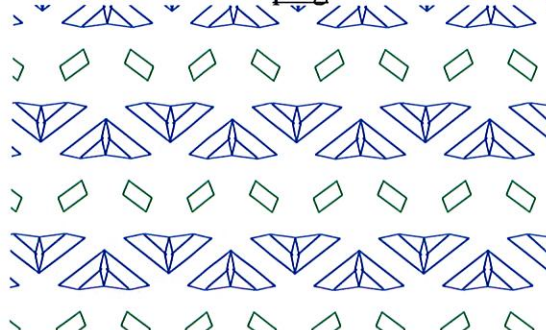
13. <u>pm</u>

14. <u>cm</u>

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15. <u>pg</u>

16. <u>pgg</u>

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8
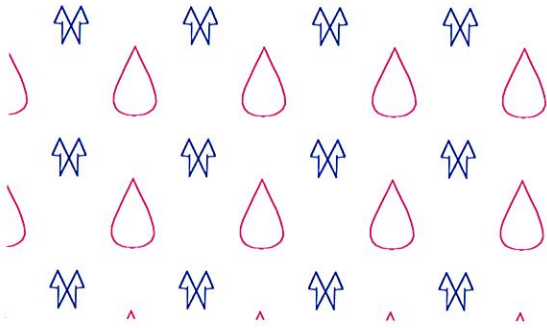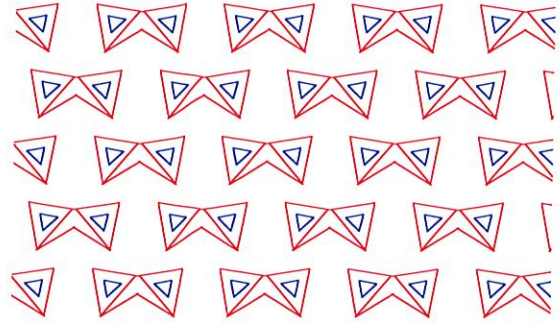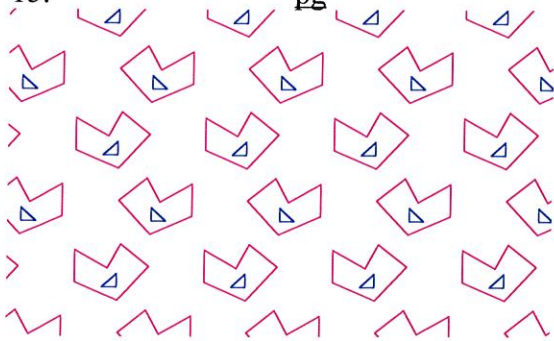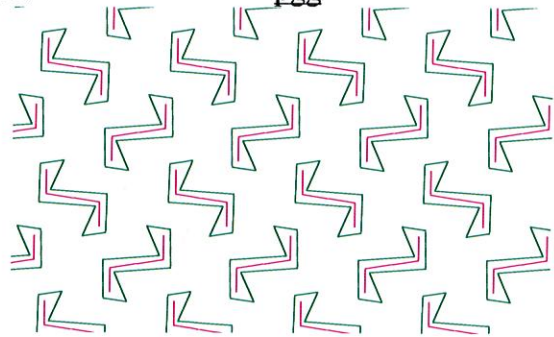
**17.** _____**p1**_____

MATH ROCKS!   MATH ROCKS!   MATH ROCKS!   MATH ROCKS!   M

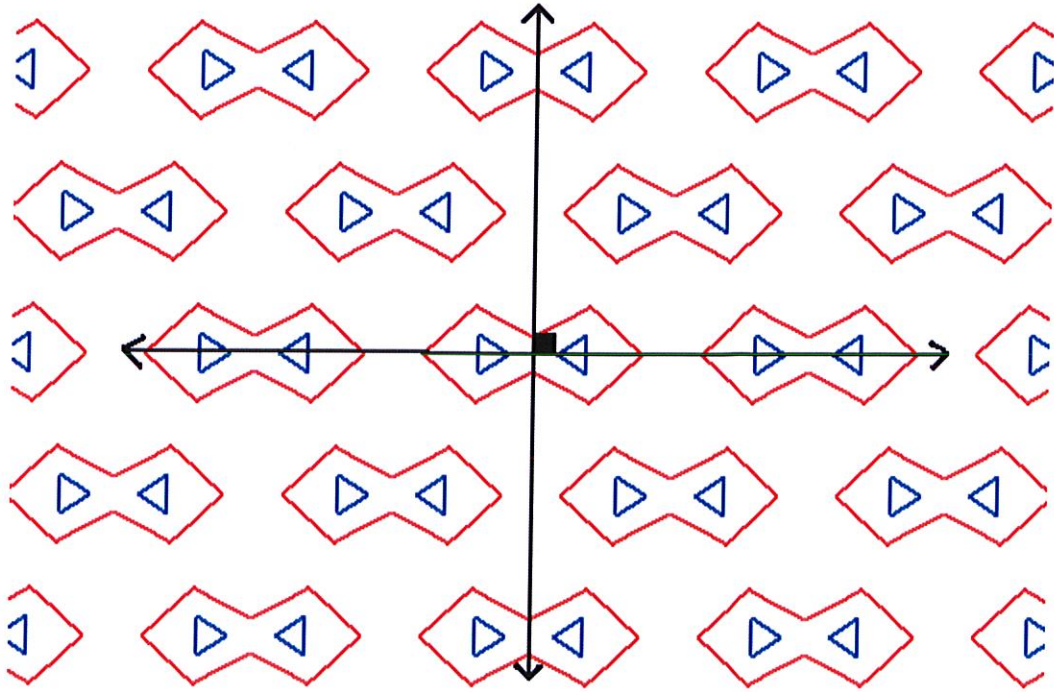MATH ROCKS!   MATH ROCKS!   MATH ROCKS!   MATH ROCKS!   M

MATH ROCKS!   MATH ROCKS!   MATH ROCKS!   MATH ROCKS!   M

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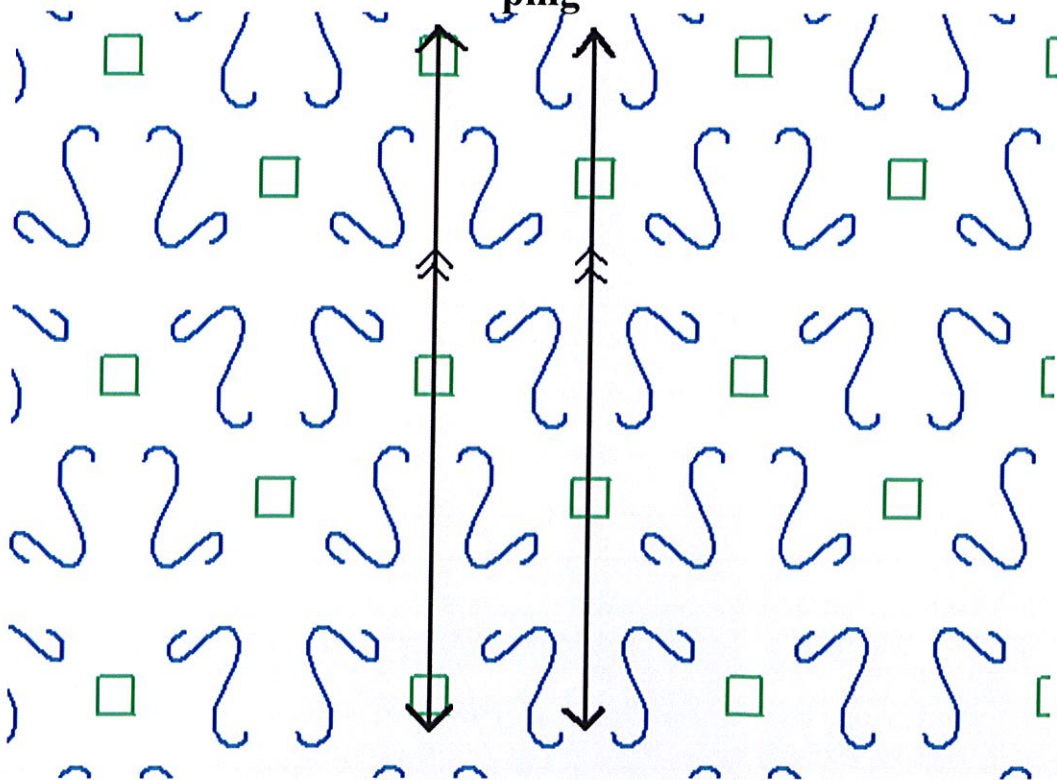
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# Answer Key for Worksheet 1

1. **p2** - translation, rotation (order 2)

2. **pmm** - translation, reflection, rotation (order 2)

3. **p3** - translation, rotation (order 3)

4. **p3m1** - translation, reflection, glide reflection, rotation (order 3)

5. **p4** - translation, rotation (order 4)

6. **p4m** - translation, reflection, glide reflection, rotation (order 4)

7. **p6** - translation, rotation (order 6)

8. **p6m** - translation, reflection, glide reflection, rotation (order 6)

9. **p4g** - translation, reflection, glide reflection, rotation (order 4)

10. **p31m** - translation, reflection, glide reflection, rotation (order 3)

11. **cmm** - translation, reflection, glide reflection, rotation (order 2)

12. **pmg** - translation, reflection, glide reflection, rotation (order 2)

13. **pm** - translation, reflection

14. **cm** - translation, reflection, glide reflection

15. **pg** - translation, glide reflection

16. **pgg** - translation, glide reflection, rotation (order 2)

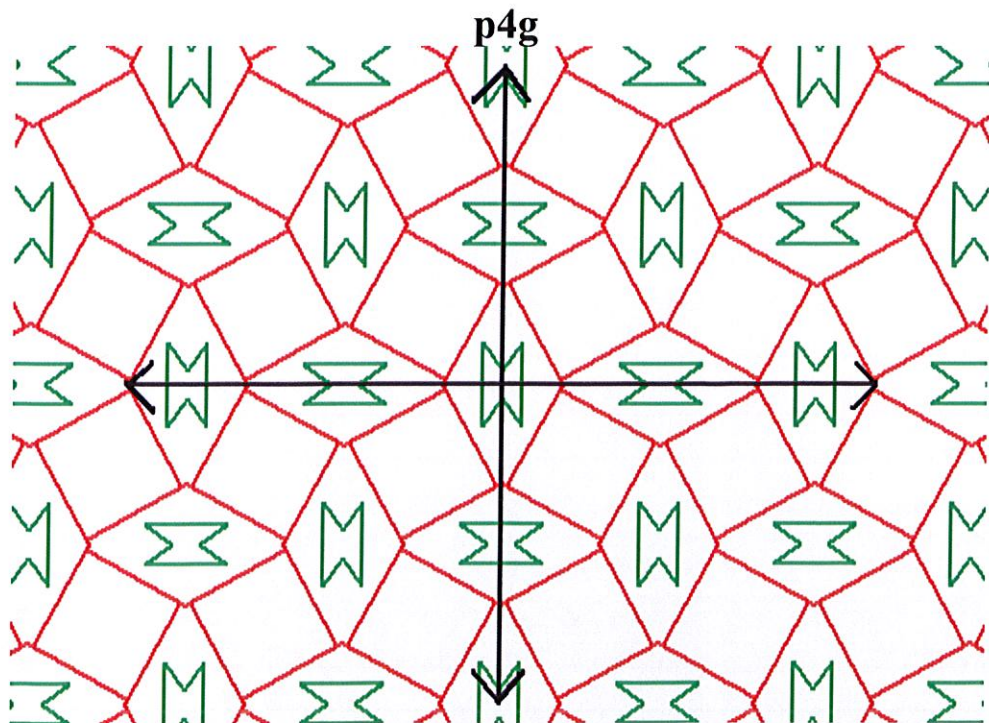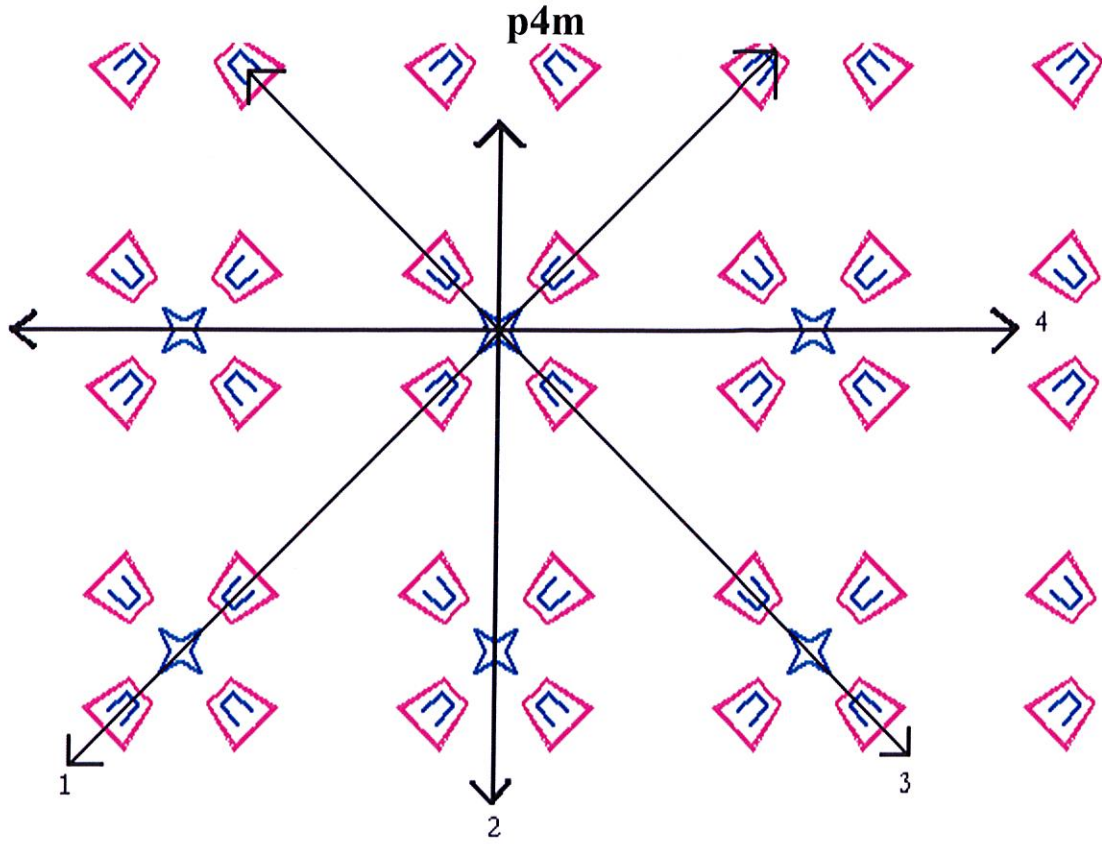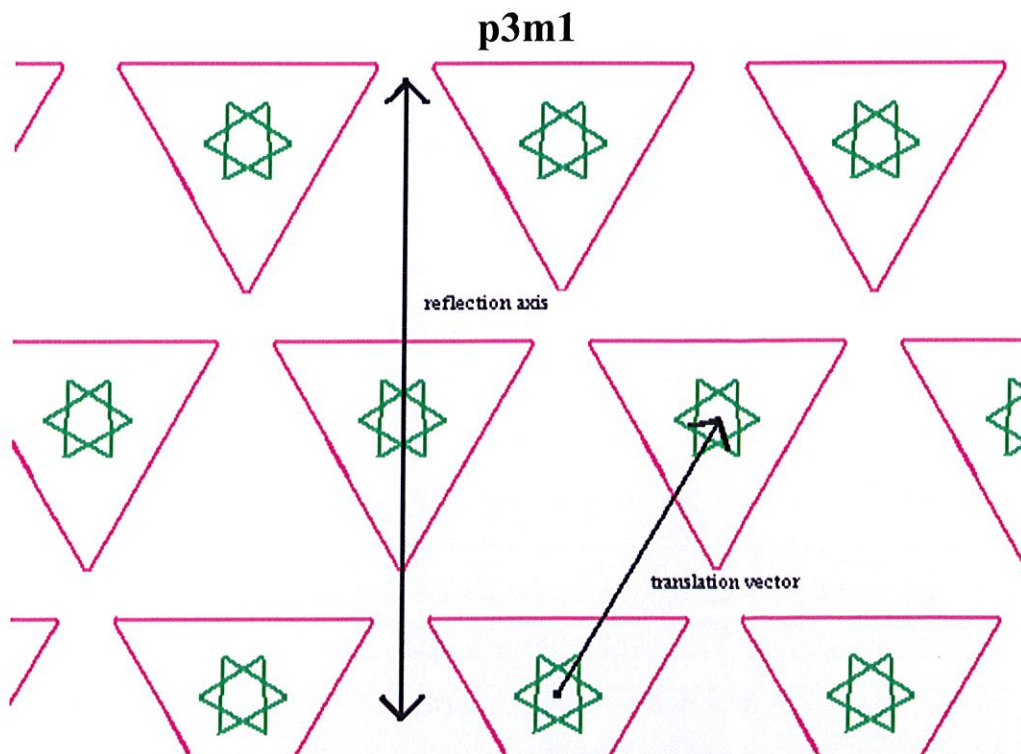17. **p1** - translation

Comparison of cmm and pmg

## cmm



## pmg
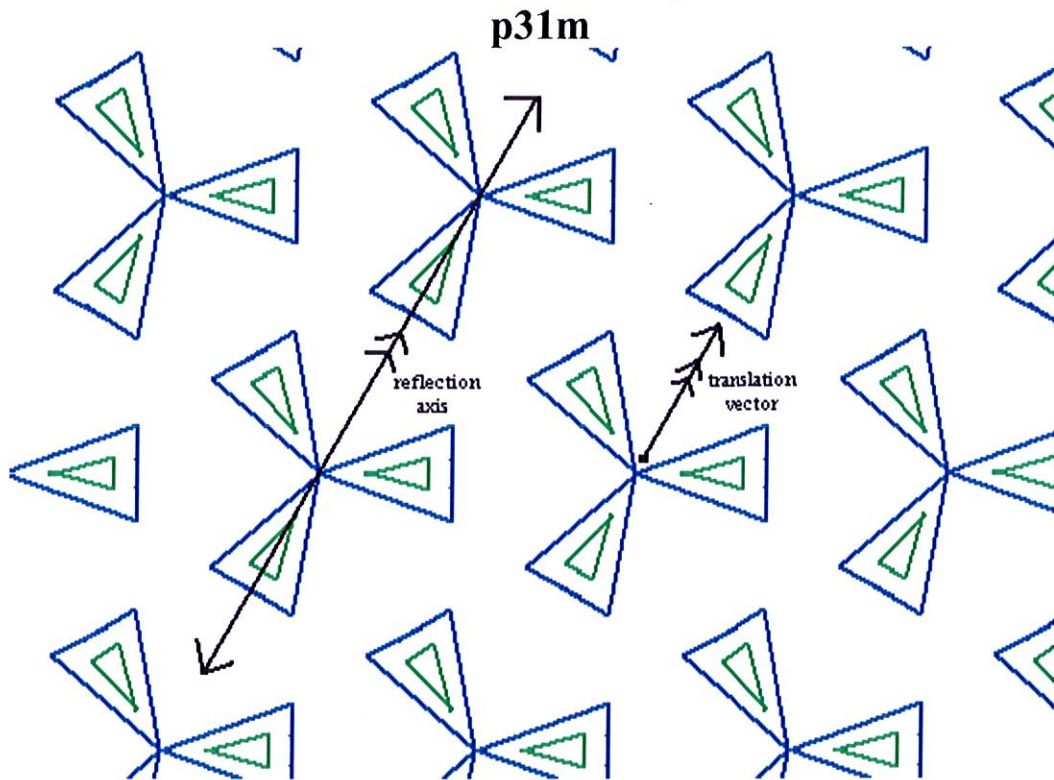
Comparison of p4m and p4g

# p4m



# p4g

Comparison of p31m and p3m1

# p31m



reflection
axis

translation
vector

# p3m1



reflection axis

translation vector

# Blank Chart for Wallpaper Group Properties

| TYPE | TRANSLATION | REFLECTION | ROTATION | ORDER OF ROTATION | GLIDE REFLECTION | SPECIAL CHARACTERISTIC |
|------|-------------|------------|----------|-------------------|------------------|------------------------|
| p1   |             |            |          |                   |                  |                        |
| p2   |             |            |          |                   |                  |                        |
| Pm   |             |            |          |                   |                  |                        |
| Pg   |             |            |          |                   |                  |                        |
| Cm   |             |            |          |                   |                  |                        |
| Pmm  |             |            |          |                   |                  |                        |
| Pmg  |             |            |          |                   |                  |                        |
| Pgg  |             |            |          |                   |                  |                        |
| Cmm  |             |            |          |                   |                  |                        |
| p4   |             |            |          |                   |                  |                        |
| p4m  |             |            |          |                   |                  |                        |
| p4g  |             |            |          |                   |                  |                        |
| p3   |             |            |          |                   |                  |                        |
| p31m |             |            |          |                   |                  |                        |
| p3m1 |             |            |          |                   |                  |                        |
| p6   |             |            |          |                   |                  |                        |
| p6m  |             |            |          |                   |                  |                        |

# Completed Chart of Wallpaper Group Properties

| TYPE | TRANSLATION | REFLECTION | ROTATION | ORDER OF ROTATION | GLIDE REFLECTION | SPECIAL CHARACTERISTIC |
|---|---|---|---|---|---|---|
| p1 | Yes | No | No | NA | No | |
| p2 | Yes | No | Yes | 2 | No | |
| Pm | Yes | Yes | No | NA | No | |
| Pg | Yes | No | No | NA | Yes | |
| Cm | Yes | Yes | No | NA | Yes | |
| Pmm | Yes | Yes | Yes | 2 | No | |
| Pmg | Yes | Yes | Yes | 2 | Yes | Reflection axes are parallel. |
| Pgg | Yes | No | Yes | 2 | Yes | |
| Cmm | Yes | Yes | Yes | 2 | Yes | Reflection axes are perpendicular. |
| p4 | Yes | No | Yes | 4 | No | |
| p4m | Yes | Yes | Yes | 4 | Yes | Four reflection axes. |
| p4g | Yes | Yes | Yes | 4 | Yes | Two reflection axes. |
| p3 | Yes | No | Yes | 3 | No | |
| p31m | Yes | Yes | Yes | 3 | Yes | Translations are parallel to reflection axes. |
| p3m1 | Yes | Yes | Yes | 3 | Yes | Translations are not parallel to reflection axes. |
| p6 | Yes | No | Yes | 6 | No | |
| p6m | Yes | Yes | Yes | 6 | Yes | |

Worksheet 2: Wallpaper Groups – Identifying the 17 types

Name_____

Identify the following patterns as one of the Wallpaper Groups listed in the word bank below.

| p1 | p2 | pm | pg | cm | pmm | pmg | pgg | |
|----|----|----|----|----|-----|-----|-----|---|
| cmm | p4 | p4m | p4g | p3 | p31m | p3m1 | p6 | p6m |

1.

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17.

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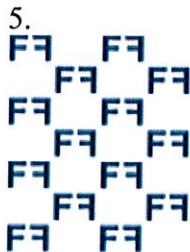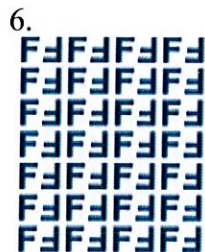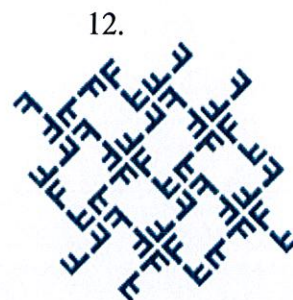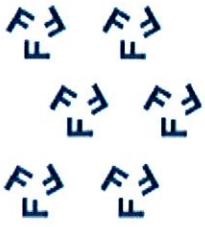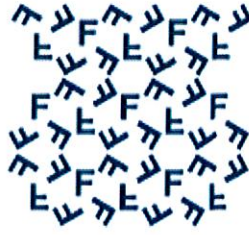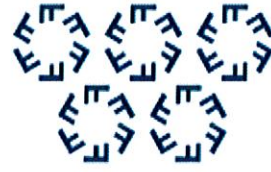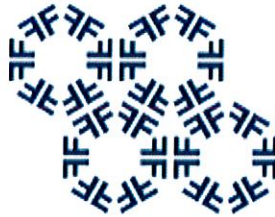
Answer Key for Worksheet 2

1. p1

2. pm

3. pmm

4. pg

5. cm

6. p2

7. pmg

8. pgg

9. cmm

10. p4

11. p4m

12. p4g

13. p3

14. p31m

15. p3m1

16. p6

17. p6m

# Drawing Symmetric Figures



rotation

reflection

translation
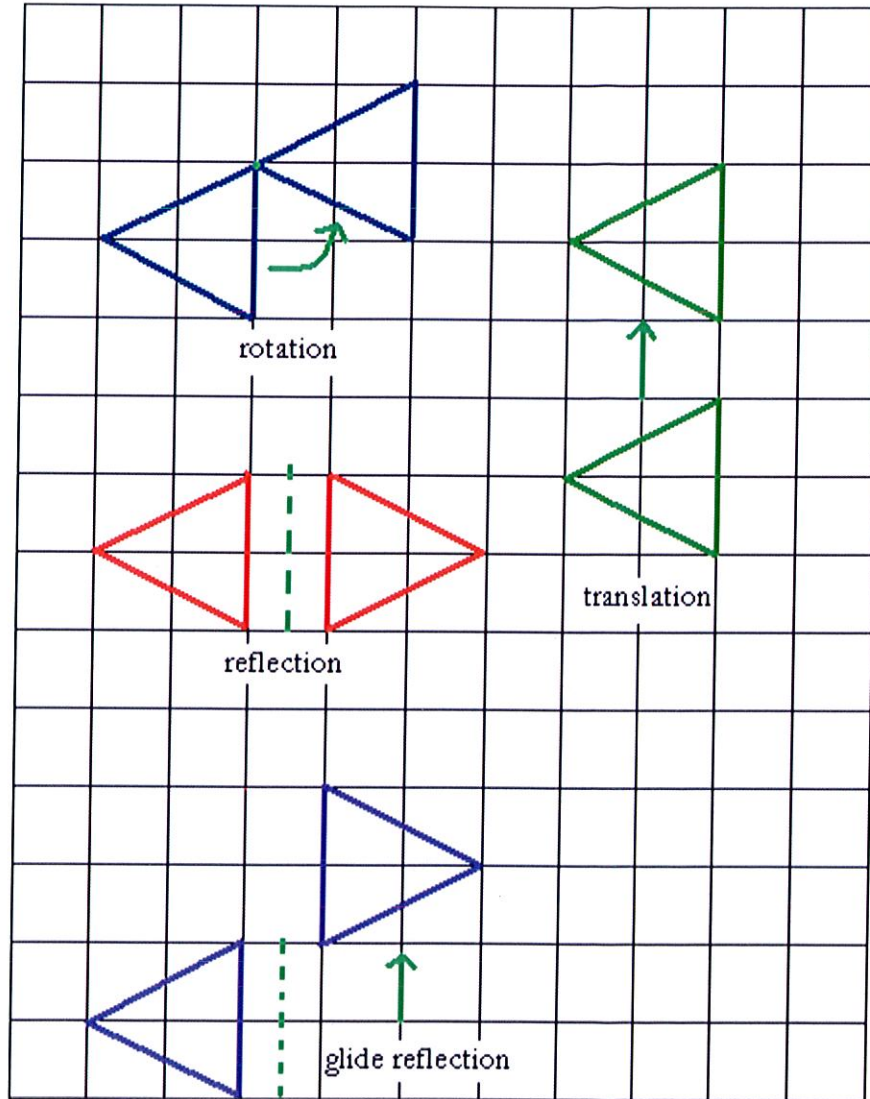
glide reflection

# Worksheet 3: Drawing Symmetries

Name_____

1. Reflect over the dotted line.
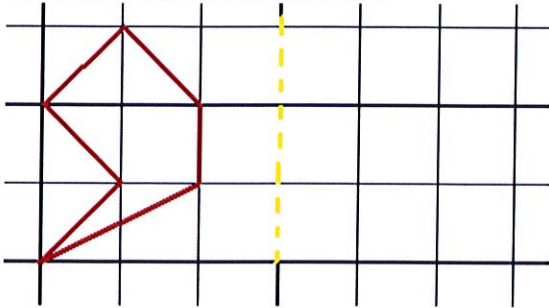
2. Translate 4 squares to the right.
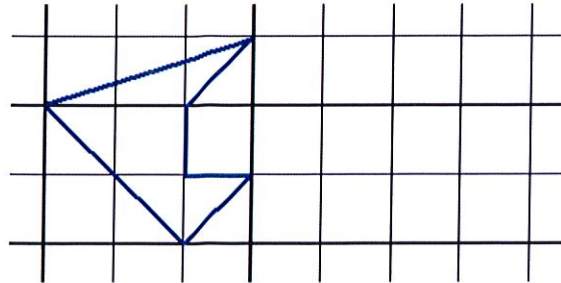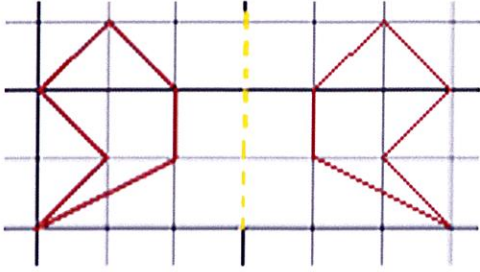
3. Rotate 90 degrees counterclockwise around point A.
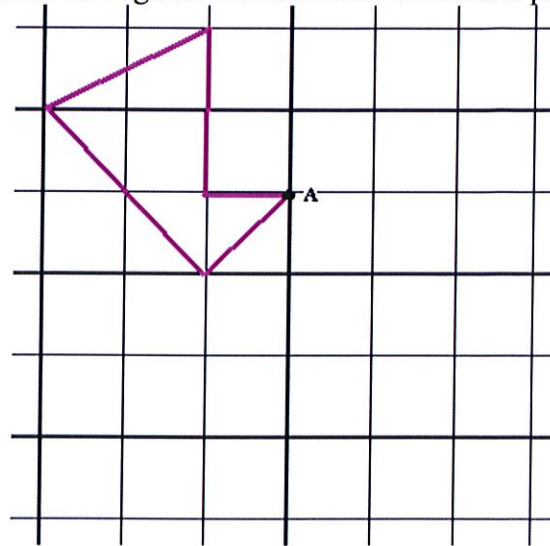
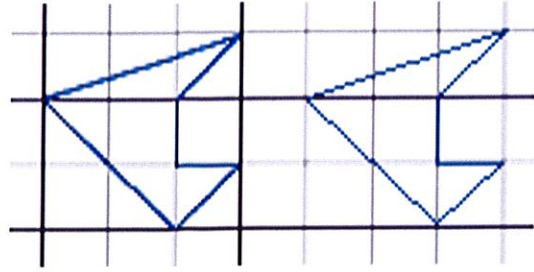4. Glide Reflection – Reflect over the dotted line, Translate 3 squares to the right.
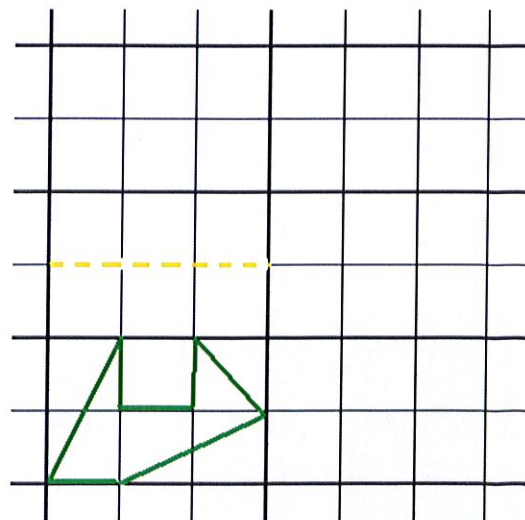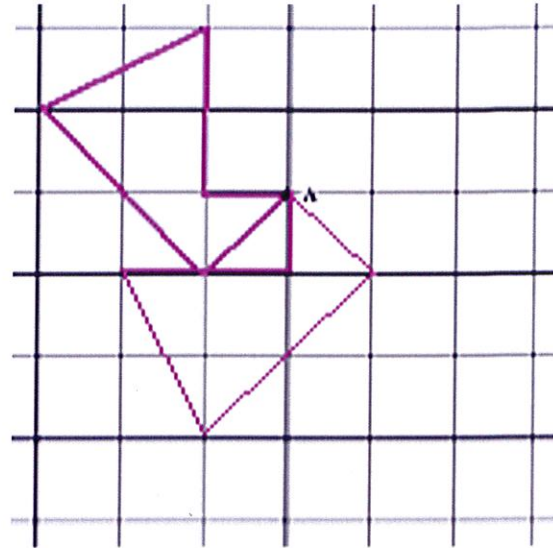
# Answer Key for Worksheet 3

1.

2.

3.

4.

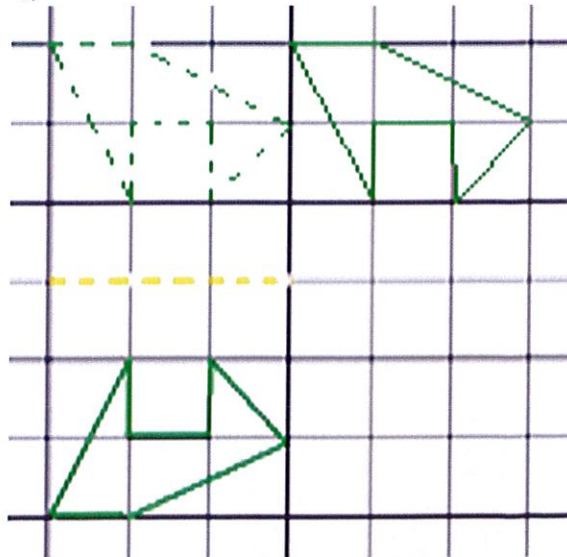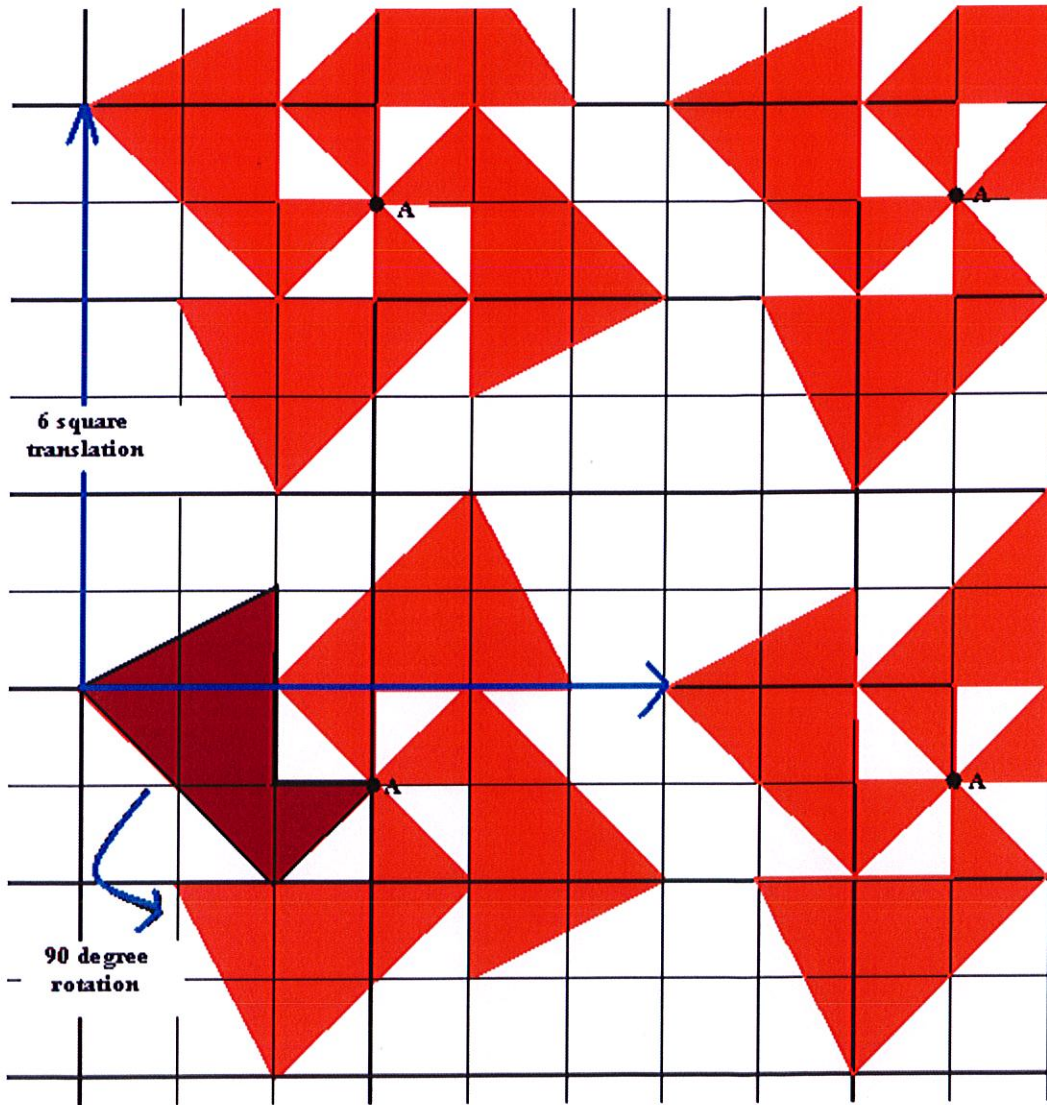Example of Creating Your Own Design



6 square
translation

90 degree
rotation

# Conversion Chart for Wallpaper Groups

| Notation Used in Kali | Our Notation |
|:---:|:---:|
| o | p1 |
| 2222 | p2 |
| ** | pm |
| xx | pg |
| *2222 | pmm |
| 22* | pmg |
| 22x | pgg |
| x* | cm |
| 2*22 | cmm |
| 442 | p4 |
| *442 | p4m |
| 4*2 | p4g |
| 333 | p3 |
| *333 | p3m1 |
| 3*3 | p31m |
| 632 | p6 |
| *632 | p6m |